THE EXPECTED DISCOUNT FACTOR DETERMINED FOR PRESENT VALUE GIVEN AS ORDERED FUZZY NUMBER

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Abstract: *The starting point for our discussion is the present value (PV) determined by means of positive fuzzy number. The information described by the so-determined PV may be supplemented with a subjective forecast of the sense trend of observed current market price. This forecast is implemented in the supplemented PV model, as the orientation of fuzzy number. The prediction of rise in market price is described as a positive orientation of ordered fuzzy number. In analogous way, the prediction of fall market price is described as the negative orientation of ordered fuzzy number. In this way, PV is presented as ordered fuzzy number. So specified PV is used for determine the simple return rate. With the assumption that the FV is a random variable under Gaussian distribution of probability, discount factor is described as ordered fuzzy random variable. At the end it is shown that the expected discount factor is determined as ordered fuzzy number. The orientation of expected discount factor is consistent with orientation of the PV defining it.*

Key words: *present value, ordered fuzzy number, discount factor*

JEL codes: *C44, C02, G10*

1. Introduction

By a security we understand an authorization to receive future financial revenue, payable to a certain maturity. The value of this revenue is interpreted as anticipated future value (FV) of the asset. In Piasecki and Siwek (2017) it is detailed justified that FV is a random variable.

In general, any security may be evaluated by means of some function of FV and the present value (PV) defined as a present equivalent of a cash flow in a given time in the present or future (Piasecki, 2012). It is commonly accepted that the PV of a future cash flow can be imprecise. The natural consequence of this approach is estimating PV with fuzzy numbers. A detailed description of the evolution of this particular model can be found in Piasecki (2014).

In Piasecki (2011) and Piasecki and Siwek (2015) the behavioural present value (BPV) was defined as such approximation of current market price which is imprecisely estimated under impact of behavioural factors. In (Łyczkowska, 2017) the information described by BPV is supplemented with a subjective forecast of the market price trend. This forecast was implemented in the model BPV as an orientation of fuzzy number. In this way the BPV was replaced by oriented BPV described by an ordered fuzzy number (Kosiński et al., 2003). The positive orientation of fuzzy number describes a subjective prediction of rise in market price. The negative orientation of fuzzy number describes a subjective prediction of fall in market price.

The main tool of a security assessment is return rate defined as any nonincreasing function of PV and nondecreasing one of FV. For the case of security with oriented PV estimated by ordered fuzzy number the expected return rate is determined in Piasecki (2017). In Piasecki and Siwek (2017) it is shown that, for appraising the considered securities, the expected fuzzy discount factor is better tool for portfolio analysis than expected fuzzy return rate.

Therefore the main purpose of presented article is to determine expected discount factor for the case when the security is evaluated by oriented PV. For this discount factor its ambiguity index will be proposed. The results obtained in this way should facilitate the analysis of the portfolio of securities with oriented PV.

2. Elements of ordered fuzzy number theory

By $\mathcal{F}(\mathbb{R})$ we denote the family of all fuzzy subsets of a real line $\mathbb{R}$. An imprecise number is a family of values in which each considered value belongs to it in a varying degree. A commonly accepted model of imprecise number is the fuzzy number, defined as a fuzzy subset of the real line $\mathbb{R}$. The most general definition of fuzzy number is given as follows:

Definition 1: (Dubois and Prade, 1979): The fuzzy number (FN) is a fuzzy subset $\mathcal{L} \in \mathcal{F}(\mathbb{R})$ with bounded support:

$$\mathbb{S}(\mathcal{L}) = \{x \in \mathbb{R}: \mu_{\mathcal{L}}(x) > 0\}, \tag{1}$$

and represented by its semi-continuous from above membership function $\mu_{\mathcal{L}} \in [0;1]^{\mathbb{R}}$ satisfying the conditions:

$$\exists_{x\in\mathbb{S}(\mathcal{L})} \ \mu_{\mathcal{L}}(x) = 1, \tag{2}$$

$$\forall_{(x,y,z)\in(\mathbb{S}(\mathcal{L}))^3} \ x \leq y \leq z \Longrightarrow \mu_{\mathcal{L}}(y) \geq \min\{\mu_{\mathcal{L}}(x); \mu_{\mathcal{L}}(z)\}. \square \tag{3}$$

The set of all FN we denote by the symbol $\mathbb{F}$. Dubois and Prade (1978) first introduced the arithmetic operations on FN. These arithmetic operations are coherent with the Zadeh Extension Principle (Zadeh, 1975a, 1975b, 1975c). Among other things, Dubois and Prade (1980) have distinguished a special type of representation of FN called LR-type FN which may be generalized in following way.

Definition 2: Let for any nondecreasing sequence $\{a,b,c,d\} \subset \mathbb{R}$ the left reference function $L_{\mathcal{L}} \in [0;1]^{[a,b]}$ and the right reference function $R_{\mathcal{L}} \in [0;1]^{[c,d]}$ are continuous from above monotonic functions satisfying the condition:

$$L_{\mathcal{L}}(b) = R_{\mathcal{L}}(c) = 1. \tag{4}$$

Then the identity:

$$\mu_{FN}(x|a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}}) = \begin{cases} 0, & x \notin [a,d] = [d,a], \\ L_{\mathcal{L}}(x), & x \in [a,b] = [b,a], \\ 1, & x \in [b,c] = [c,b], \\ R_{\mathcal{L}}(x), & x \in [c,d] = [d,c] \end{cases} \tag{5}$$ [1]

defines the membership function $\mu_{FN}(\cdot\,|a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}}) \in [0,1]^{\mathbb{R}}$ of the FN $\mathcal{L}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$ which is called LR-type FN (LR-FN).$\square$

For any LR-FN $\mathcal{L}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$ we have:

$$]a,d[\subset \mathbb{S}(\mathcal{L}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})) \subset [a,d]. \tag{6}$$

In Goetschel and Voxman (1986) it is proved that any FN may be described as LR-FN in the sense given by the Definition 2. The following terms are applied in this proof.

Definition 3: Pseudo inverse function $l^{\lhd} \in [0;1]^{[l(0),l(1)]}$ of any bounded continuous and nondecreasing function $l \in [l(0),l(1)]^{[0;1]}$ is given by the identity:

$$l^{\lhd}(x) = \max\{\alpha \in [0;1]: l(\alpha) = x\}. \tag{7}$$

Definition 4: Pseudo inverse function $r^{\lhd} \in [0;1]^{[r(1),r(0)]}$ of any bounded continuous and nonincreasing function $r \in [r(0),r(1)]^{[0;1]}$ is given by the identity:

$$r^{\lhd}(x) = \min\{\alpha \in [0;1]: r(\alpha) = x\}. \tag{8}$$

The concept of ordered fuzzy numbers (OFN) was introduced by Kosiński and his co-writers in the series of papers (Kosiński et al., 2002; Kosiński, 2006) as an extension of the concept of FN. Thus, any OFN should be determined as a fuzzy subset in the real line $\mathbb{R}$. On the other hand, Kosiński has defined OFN as a ordered pair of

[1] Let us note that this identity describes additionally extended notation of numerical intervals, which is used in this work.

functions from the unit interval $[0,1]$ into $\mathbb{R}$. This kind of pair is not a fuzzy subset in $\mathbb{R}$. Thus we can not accept original Kosiński's terminology. What is more, the intuitive Kosiński's approach to the notion of OFN is very useful. For these reasons, below we present a revised definition of OFN which fully corresponds to the intuitive definition by Kosiński. The OFN concept of a number is closely linked to the following ordered pair.

Definition 5. By the Kosiński's pair $[\![\mathcal{L}]\!]$ we understand the ordered pair $(f_{\mathcal{L}}, g_{\mathcal{L}})$ of monotonic continuous surjections $f_{\mathcal{L}}: [0,1] \to UP_{\mathcal{L}} = [f_{\mathcal{L}}(0), f_{\mathcal{L}}(1)]$ and $g_{\mathcal{L}}: [0,1] \to DOWN_{\mathcal{L}} = [g_{\mathcal{L}}(0), g_{\mathcal{L}}(1)]$ fulfilling the conditions:

$$\big(f_{\mathcal{L}}(1) - f_{\mathcal{L}}(0)\big) \cdot \big(g_{\mathcal{L}}(1) - g_{\mathcal{L}}(0)\big) \le 0, \tag{9}$$

$$|f_{\mathcal{L}}(1) - g_{\mathcal{L}}(1)| \le |f_{\mathcal{L}}(0) - g_{\mathcal{L}}(0)|, \tag{10}$$

$$UP_{\mathcal{L}} \cap DOWN_{\mathcal{L}} = \{f_{\mathcal{L}}(1)\} \cap \{g_{\mathcal{L}}(1)\}. \; \square \tag{11}$$

Remark: In the original final version of the Kosiński's definition (Kosiński, 2006), the OFN is defined as an ordered pair $(f_{\mathcal{L}}, g_{\mathcal{L}})$ of continuous functions $f_{\mathcal{L}}: [0,1] \to UP_{\mathcal{L}}$ and $g_{\mathcal{L}}: [0,1] \to DOWN_{\mathcal{L}}$. Kosiński marked other conditions for the above definition on the graphs only. Because OFN defined this way is not a fuzzy set, in the Definition 5 the Kosiński's term OFN was replaced by the term "Kosiński's pair". □

For any Kosiński's pair $(f_{\mathcal{L}}, g_{\mathcal{L}})$ the function $f_{\mathcal{L}}: [0,1] \to UP_{\mathcal{L}}$ is called the up-function. Then the function $g_{\mathcal{L}}: [0,1] \to DOWN_{\mathcal{L}}$ is called down-function. The up-function and down-function are collectively referred as Kosiński's maps. The condition (9) implies that Kosiński's maps cannot be increasing or decreasing at the same time. Knowing this fact, we define OFN in following way.

Definition 6. For fixed Kosiński's pair $[\![\mathcal{L}]\!]$ the OFN $\overleftrightarrow{\mathcal{L}}$ is defined as the pair $(\mathcal{L}, \mho)$ of LR-FN $\mathcal{L} \in \mathbb{F}$ and orientation $\mho$ in that the following way:

- the left reference function $L_{\mathcal{L}}$ is equal to pseudo-inverse function of the nondecreasing Kosiński's map;
- the right reference function $R_{\mathcal{L}}$ is equal to pseudo-inverse function of the nonincreasing Kosiński's map;
- orientation $\mho$ is determined as common sense of all vectors from up-function range $UP_{\mathcal{L}}$ to down-function range $DOWN_{\mathcal{L}}$. □

The above definition is coherent to the intuitive Kosiński's approach to the OFN term. Therefore we agree with other scientists that the OFN should be called the Kosiński's number (Prokopowicz and Pedrycz, 2015). The space of all OFN is denoted by the symbol $\mathbb{K}$. For any OFN $\overleftrightarrow{\mathcal{L}} \in \mathbb{K}$, its up-function is denoted by $f_{\mathcal{L}}$ and its down-function is denoted by $g_{\mathcal{L}}$. The OFN $\overleftrightarrow{\mathcal{L}} \in \mathbb{K}$ is explicitly determined by its membership function $\mu_{\overrightarrow{OFN}}(\cdot\,|f_{\mathcal{L}}(0), f_{\mathcal{L}}(1), g_{\mathcal{L}}(1), g_{\mathcal{L}}(0), f_{\mathcal{L}}^{\triangleleft}, g_{\mathcal{L}}^{\triangleleft}) \in [0,1]^{\mathbb{R}}$ given as follows:

$$\mu_{\overrightarrow{OFN}}(x|f_{\mathcal{L}}(0), f_{\mathcal{L}}(1), g_{\mathcal{L}}(1), g_{\mathcal{L}}(0), f_{\mathcal{L}}^{\triangleleft}, g_{\mathcal{L}}^{\triangleleft}) = \begin{cases} 0, & x \notin [f_{\mathcal{L}}(0), g_{\mathcal{L}}(0)] = [g_{\mathcal{L}}(0), f_{\mathcal{L}}(0)], \\ f_{\mathcal{L}}^{\triangleleft}(x), & x \in [f_{\mathcal{L}}(0), f_{\mathcal{L}}(1)] = [f_{\mathcal{L}}(1), f_{\mathcal{L}}(0)], \\ 1, & x \in [f_{\mathcal{L}}(1), g_{\mathcal{L}}(1)] = [g_{\mathcal{L}}(1), f_{\mathcal{L}}(1)], \\ g_{\mathcal{L}}^{\triangleleft}(x), & x \in [g_{\mathcal{L}}(1), g_{\mathcal{L}}(0)] = [g_{\mathcal{L}}(0), g_{\mathcal{L}}(1)]. \end{cases} \tag{12}$$

The OFN $\overleftrightarrow{\mathcal{L}} \in \mathbb{K}$ determined by the membership function $\mu_{\overrightarrow{OFN}}(\cdot\,|f_{\mathcal{L}}(0), f_{\mathcal{L}}(1), g_{\mathcal{L}}(1), g_{\mathcal{L}}(0), f_{\mathcal{L}}^{\triangleleft}, g_{\mathcal{L}}^{\triangleleft}) \in [0,1]^{\mathbb{R}}$ we will denote by the symbol $\overleftrightarrow{\mathcal{L}}(f_{\mathcal{L}}(0), f_{\mathcal{L}}(1), g_{\mathcal{L}}(1), g_{\mathcal{L}}(0), f_{\mathcal{L}}^{\triangleleft}, g_{\mathcal{L}}^{\triangleleft})$. Taking into account all above considerations, we can to define equivalently OFN in following way.

Definition 7: Let for any nondecreasing sequence $\{a, b, c, d\} \subset \mathbb{R}$ the starting-function $S_{\mathcal{L}}: [a, b] \longrightarrow [0,1]$ and the ending-function $E_{\mathcal{L}}: [c, d] \longrightarrow [0,1]$ are continuous from above monotonic functions satisfying the condition:

$$S_{\mathcal{L}}(b) = E_{\mathcal{L}}(c) = 1. \tag{13}$$

Then the identity:

$$\mu_{\overrightarrow{OFN}}(x|a, b, c, d, S_{\mathcal{L}}, E_{\mathcal{L}}) = \begin{cases} 0, & x \notin [a, d] = [d, a], \\ S_{\mathcal{L}}(x), & x \in [a, b] = [b, a], \\ 1, & x \in [b, c] = [c, b], \\ E_{\mathcal{L}}(x), & x \in [c, d] = [d, c] \end{cases} \tag{14}$$

defines the membership function $\mu_{\overrightarrow{OFN}}(\cdot\,|a, b, c, d, S_{\mathcal{L}}, E_{\mathcal{L}}) \in [0,1]^{\mathbb{R}}$ of the OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_{\mathcal{L}}, E_{\mathcal{L}})$. □

Moreover, we have here:

$$]a,d[\subset \mathbb{S}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})\right)\subset[a,d] \tag{15}$$

The conditions (9), (10) and (11) imply that this OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})$ fulfils exactly one from the following conditions:

$$a\le b\le c\le d, \tag{16}$$

$$a\ge b\ge c\ge d. \tag{17}$$

The condition (16) describes the positive orientation $\mho$ of OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})$. In this case, the starting-function $S_{\mathcal{L}}$ is nondecreasing and the ending-function $E_{\mathcal{L}}$ is nonincreasing. The space of all positive oriented OFN we denote by the symbol $\mathbb{K}^+$. Any positive oriented OFN is interpreted as such imprecise number, which may increase.

The condition (17) describes negative orientation $\mho$ of OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})$. In this case, the starting-function $S_{\mathcal{L}}$ is nonincreasing and the ending-function $E_{\mathcal{L}}$ is nondecreasing. The space of all negative oriented OFN we denote by the symbol $\mathbb{K}^-$. Negative oriented OFN is interpreted as such imprecise number, which may decrease.

Arithmetic operations on OFN are defined by Kosiński (2006) as extension of arithmetic operations on the real numbers. In a special case, for any Kosiński's pair $[\![\mathcal{L}]\!]=(f_{\mathcal{L}},g_{\mathcal{L}})$ determining the OFN $\overleftrightarrow{\mathcal{L}}=\overleftrightarrow{\mathcal{L}}(a,b,c,d,f_{\mathcal{L}}^{\lhd},g_{\mathcal{L}}^{\lhd})$ and for any monotonic function $h:\mathbb{R}\supset\mathbb{A}\to\mathbb{R}$ we have:

$$h(\overleftrightarrow{\mathcal{L}})=\overleftrightarrow{Z}=\overleftrightarrow{Z}\,(h(a),h(b),h(c),h(d),(h\circ f_{\mathcal{L}})^{\lhd},(h\circ g_{\mathcal{L}})^{\lhd}). \tag{18}$$

At the end, let us note that for any for any Kosiński's pair $[\![\mathcal{L}]\!]$ determining the OFN $\overleftrightarrow{\mathcal{L}}=\overleftrightarrow{\mathcal{L}}(a,b,c,d,f_{\mathcal{L}}^{\lhd},g_{\mathcal{L}}^{\lhd})$ and for any $\gamma\neq 0$ we have the identity:

$$\mu_{\overrightarrow{OFN}}(\gamma\cdot x|a,b,c,d,f_{\mathcal{L}}^{\lhd},g_{\mathcal{L}}^{\lhd})=\mu_{\overrightarrow{OFN}}\left(x\middle|\frac{a}{\gamma},\frac{b}{\gamma},\frac{c}{\gamma},\frac{d}{\gamma},(d(\cdot|\gamma)^{\circ}f_{\mathcal{L}})^{\lhd},(d(\cdot|\gamma)^{\circ}g_{\mathcal{L}})^{\lhd}\right), \tag{19}$$

where the function $d(\cdot|\gamma)\in\mathbb{R}^{\mathbb{R}}$ is given by the identity:

$$d(x|\gamma)=\frac{x}{\gamma}. \tag{20}$$

Ambiguity of OFN is interpreted as a lack of a clear recommendation one value between among various others. An increase in information ambiguity makes it less useful and therefore it is logical to consider the problem of ambiguity assessment. In Piasecki and Siwek (2017) the ambiguity of LR-FN $\mathcal{L}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$ is evaluated by energy measure $e\in[\mathbb{R}_0^+]^{\mathbb{F}}$ determined as integral of its membership function $\mu_{FN}(\cdot|a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$ in following way:

$$e\big(\mathcal{L}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})\big)=\int_a^d\mu_{FN}(x|a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})dx. \tag{21}$$

Thus we propose to apply for ambiguity evaluation the ambiguity index $a\in\mathbb{R}^{\mathbb{K}}$ which assets the ambiguity of OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})$ by integral of its membership function $\mu_{\overrightarrow{OFN}}(\cdot|a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})$ as follows:

$$a\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})\right)=\int_a^d\mu_{\overrightarrow{OFN}}(x|a,b,c,d,S_{\mathcal{L}},E_{\mathcal{L}})dx. \tag{22}$$

Quite new for evaluation fact is that for any negative oriented OFN its ambiguity index is negative. This gives new perspectives for portfolio risk management.

4. Oriented fuzzy present value

Let us consider the fixed security. We observe a market price $\check{C}>0$ of this security. In line with the assumption applied in Piasecki (2011b, 2017) and Piasecki and Siwek (2015, 2017), PV is such a positive fuzzy number, which is an approximation of the market price $\check{C}$. Therefore, we can determine PV as positive LR-FN $Pv(\check{C}_{min},\check{C}_*,\check{C}^*,\check{C}_{max},L_{Pv},R_{Pv})$, where:

- $\check{C}_{min}\in\left]0,\check{C}\right]$ is the maximal lower bound of PV,
- $\check{C}_{max}\left[\check{C},+\infty\right[$ is the minimal upper bound of PV,
- $\check{C}_*\in\left[\check{C}_{min},\check{C}\right]$ is the minimal upper assessment of prices visibly lower than the market price $\check{C}$,
- $\check{C}^*\in\left[\check{C},\check{C}_{max}\right]$ is the maximal lower assessment of prices visibly higher than the market price $\check{C}$,
- (L_{Pv},R_{Pv}) is any ordered pair of references functions described in the Definition 2.

The method of determining parameters $\check{C}_{min}, \check{C}_{max}$ is given in Piasecki and Siwek (2015). For given left reference function $L_{Pv} \in [0,1]^{[\check{C}_{min}, \check{C}_*]}$ and right reference function $R_{Pv} \in [0,1]^{[\check{C}^*, \check{C}_{max}]}$, the membership function $\mu_{Pv} \in [0,1]^{\mathbb{R}}$ of LR-FN $Pv(\check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}, L_{Pv}, R_{Pv})$ is uniquely defined by identity (5).

In this section the information described by defined above PV is supplemented with a subjective forecast of orientation of the market price trend. In agree with OFN interpretation, we use here the following rules recording the alleged orientation of the trend:

- The prediction of rise in market price is described as a positive orientation of ordered fuzzy number. Then PV will be presented as OFN $\overleftrightarrow{Pv}(\check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}, L_{Pv}, R_{Pv})$.
- The prediction of fall in market price is described as the negative orientation of ordered fuzzy number. Then PV will be presented as OFN $\overleftrightarrow{Pv}(\check{C}_{max}, \check{C}_*, \check{C}^*, \check{C}_{min}, R_{Pv}, L_{Pv})$.

In this way PV is presented as the ordered fuzzy number. Each of these PV representations is called oriented fuzzy PV (OFPV). By writing OFPV with any orientation we will denote it by the symbol $\overleftrightarrow{Pv}(\check{C}_\alpha, \check{C}_\beta, \check{C}_\gamma, \check{C}_\delta, S_{Pv}, E_{Pv})$, where:

- $[\check{C}_\alpha, \check{C}_\delta] \subset \mathbb{R}^+$ is interval of all possible PV' values,
- $[\check{C}_\beta, \check{C}_\gamma] \subset [\check{C}_\alpha, \check{C}_\delta]$ is interval of all prices which differ invisible from market price $\check{C}$,
- (S_{Pv}, E_{Pv}) is any ordered pair of starting-function and ending-function described in the Definition 7.

The OFPV $\overleftrightarrow{Pv}(\check{C}_\alpha, \check{C}_\beta, \check{C}_\gamma, \check{C}_\delta, S_{Pv}, E_{Pv})$ is determined by membership function $\mu_{\overleftrightarrow{OFN}}(\cdot\,|\check{C}_\alpha, \check{C}_\beta, \check{C}_\gamma, \check{C}_\delta, S_{Pv}, E_{Pv})$ given by the identity (14). Such defined OFPV may be applied for determining the discount factor.

5. Oriented fuzzy discount factor

Let us assume that the time horizon $t > 0$ of an investment is fixed. Then, the security considered here is determined by two values: anticipated FV V_t and assessed PV V_0. The basic characteristic of benefits from owning this security is a return rate r_t given by the identity:

$$r_t = r(V_0, V_t). \tag{23}$$

In the general case, if $(V_0, V_t) \in \mathbb{R}^+ \times \mathbb{R}^+$ then the function: $r: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a decreasing function of PV and an increasing function of FV. It implies that for any pair $(V_0, V_t) \in \mathbb{R}^+ \times \mathbb{R}^+$ we can determine inverse functions $r_V^{-1}(V_0, \cdot): \mathbb{R}^+ \to \mathbb{R}^+$ and $r_0^{-1}(\cdot, V_t): \mathbb{R}^+ \to \mathbb{R}^+$. Moreover, in the special case we have here:

- simple return rate:

$$r_t = \frac{V_t - V_0}{V_0} = \frac{V_t}{V_0} - 1, \tag{24}$$

- logarithmic return rate :

$$r_t = \ln\frac{V_t}{V_0}. \tag{25}$$

In Piasecki and Siwek (2017) it is detailed justified that FV is a random variable random variable $\tilde{V}_t: \Omega \longrightarrow \mathbb{R}^+$. The set Ω is a set of elementary states ω of the financial market. In the classical approach to the problem of the return rate estimation, the security PV is identified with the observed market price $\check{C}$.Then the return rate is a random variable determined by the identity:

$$\tilde{r}_t(\omega) = r\left(\check{C}, \tilde{V}_t(\omega)\right). \tag{26}$$

In practice of financial markets analysis, the uncertainty risk is usually described by probability distribution of return rate determined by (26). At the moment, we have an extensive knowledge on this subject. Let us assume that this probability distribution is given by cumulative distribution function $F_r: \mathbb{R} \longrightarrow [0; 1]$. We assume here that the expected value $\bar{r}$ of this distribution exists. On other side, the cumulative distribution function F_r determines probability distribution $P: 2^\Omega \supset \tilde{r}^{-1}(\mathcal{B}) \longrightarrow [0; 1]$, where the symbol $\mathcal{B}$ denotes the smallest Borel σ-field containing all intervals in the real line $\mathbb{R}$. Moreover, let us note that there we have:

$$\tilde{V}_t(\omega) = r_V^{-1}\left(\check{C}, \tilde{r}_t(\omega)\right). \tag{27}$$

Let us consider now the case when PV is determined as OFPV $\overleftrightarrow{Pv}(\check{C}_\alpha, \check{C}_\beta, \check{C}_\gamma, \check{C}_\delta, S_{Pv}, E_{Pv})$ represented by its membership function $\mu_{\overleftrightarrow{Pv}} \in [0; 1]^{\mathbb{R}}$ given by the identity:

$$\mu_{\overleftrightarrow{Pv}}(x) = \mu_{\overleftrightarrow{OFN}}(x|\check{C}_\alpha, \check{C}_\beta, \check{C}_\gamma, \check{C}_\delta, S_{Pv}, E_{Pv}). \tag{28}$$

According to the Zadeh's Extension Principle, the simple return rate calculated for the OFPV is a fuzzy probabilistic set represented by its membership function $\tilde{\rho} \in [0;1]^{\mathbb{R}\times\Omega}$ given by:

$$\tilde{\rho}(r,\omega) = \sup\{\mu_{\overrightarrow{PV}}(x): r = r(x, V_t(\omega)), x \in \mathbb{R}\} = \mu_{\overrightarrow{PV}}\left(r_0^{-1}(r, V_t(\omega))\right) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(r, r_V^{-1}\left(\check{C}, \tilde{r}_t(\omega)\right)\right)\right). \quad (29)$$

Then the membership function $\rho \in [0;1]^{\mathbb{R}}$ of expected return rate is calculated in following way:

$$\rho(r) = \int_{-\infty}^{+\infty} \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(r, r_V^{-1}(\check{C}, y)\right)\right) d\, F_r(y) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(r, r_V^{-1}(\check{C}, \bar{r})\right)\right). \quad (30)$$

In Piasecki and Siwek (2017) it is shown that, for appraising the considered securities, the expected fuzzy discount factor is better tool than expected fuzzy return rate. Therefore we will to determine expected discount factor for the case of OFPV. In general, for given return rate r_t the discount factor v_t is explicitly determined by the identity:

$$r_t = r(v_t, 1). \quad (31)$$

We shall consider expected discount factor $\bar{v}$ defined by identity:

$$\bar{v} = r_0^{-1}(\bar{r}, 1). \quad (32)$$

In line with (30), the membership function $\delta \in [0,1]^{\mathbb{R}}$ of an discount factor $\mathcal{V} \in \mathbb{K}$ is given by the identity:

$$\delta(v) = \delta\left(r_0^{-1}(r,1)\right) = \rho(r) = \rho\left(r(v,1)\right) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(r(v,1), r_V^{-1}\left(\check{C}, r(\bar{v},1)\right)\right)\right). \quad (33)$$

By means of (19), for simple return rate we obtain now:

$$\delta(v) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(r(v,1), r_V^{-1}\left(\check{C}, r(\bar{v},1)\right)\right)\right) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(\frac{1}{v}-1, r_V^{-1}\left(\check{C}, \frac{1}{\bar{v}}-1\right)\right)\right) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(\frac{1}{v}-1, \frac{\check{C}}{\bar{v}}\right)\right) =$$

$$= \mu_{\overrightarrow{PV}}\left(\frac{\check{C}\cdot v}{\bar{v}}\right) = \mu_{\overrightarrow{OFN}}\left(\frac{\check{C}\cdot v}{\bar{v}}\middle|\check{C}_\alpha, \check{C}_\beta, \check{C}_\gamma, \check{C}_\delta, S_{Pv}, E_{Pv}\right) = \mu_{\overrightarrow{OFN}}\left(v\middle|\frac{\check{C}_\alpha\cdot\bar{v}}{\check{C}\cdot}, \frac{\check{C}_\beta\cdot\bar{v}}{\check{C}\cdot}, \frac{\check{C}_\gamma\cdot\bar{v}}{\check{C}\cdot}, \frac{\check{C}_\delta\cdot\bar{v}}{\check{C}\cdot}, S_{Pv}\circ d\left(\cdot\middle|\frac{\check{C}}{\bar{v}}\right), E_{Pv}\circ d\left(\cdot\middle|\frac{\check{C}}{\bar{v}}\right)\right). \quad (34)$$

For logarithmic return rate we obtain:

$$\delta(v) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(r(v,1), r_V^{-1}\left(\check{C}, r(\bar{v},1)\right)\right)\right) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(-\ln v, r_V^{-1}(\check{C}, -\ln\bar{v})\right)\right) = \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(-\ln v, \frac{\check{C}}{\bar{v}}\right)\right) =$$

$$= \mu_{\overrightarrow{PV}}\left(\frac{\check{C}\cdot v}{\bar{v}}\right) = \mu_{\overrightarrow{OFN}}\left(\frac{\check{C}\cdot v}{\bar{v}}\middle|\check{C}_\alpha, \check{C}_\beta, \check{C}_\gamma, \check{C}_\delta, S_{Pv}, E_{Pv}\right) = \mu_{\overrightarrow{OFN}}\left(v\middle|\frac{\check{C}_\alpha\cdot\bar{v}}{\check{C}}, \frac{\check{C}_\beta\cdot\bar{v}}{\check{C}}, \frac{\check{C}_\gamma\cdot\bar{v}}{\check{C}}, \frac{\check{C}_\delta\cdot\bar{v}}{\check{C}}, S_{Pv}\circ d\left(\cdot\middle|\frac{\check{C}}{\bar{v}}\right), E_{Pv}\circ d\left(\cdot\middle|\frac{\check{C}}{\bar{v}}\right)\right) \quad (35)$$

Comparison of dependence (34) and (35) raises the question: What conditions should be met in order to satisfy the below relationship for generalized return rate:

$$\delta(v) = \mu_{\overrightarrow{OFN}}\left(v\middle|\frac{\check{C}_\alpha\cdot\bar{v}}{\check{C}\cdot}, \frac{\check{C}_\beta\cdot\bar{v}}{\check{C}\cdot}, \frac{\check{C}_\gamma\cdot\bar{v}}{\check{C}\cdot}, \frac{\check{C}_\delta\cdot\bar{v}}{\check{C}\cdot}, S_{Pv}\circ d\left(\cdot\middle|\frac{\check{C}}{\bar{v}}\right), E_{Pv}\circ d\left(\cdot\middle|\frac{\check{C}}{\bar{v}}\right)\right)? \quad (36)$$

The increase in the ambiguity of an expected discount factor $\mathcal{V} \in \mathbb{K}$ leads to an increase in the number of alternative investment recommendations. It implies an increase in the risk of choosing such a financial decision, which will be burdened *ex post* by the lost profit. This kind of risk is called an ambiguity risk. The ambiguity risk burdening the expected discount factor $\mathcal{V}$ is evaluated by the absolute value of ambiguity index given by the identity:

$$|a(\mathcal{V})| = \left|\int_{\mathbb{S}(\mathcal{V})} \mu_{\overrightarrow{PV}}\left(r_0^{-1}\left(r(v,1), r_V^{-1}\left(\check{C}, r(\bar{v},1)\right)\right)\right) dv\right|. \quad (37)$$

6. Conclusions

The results of the work fully convince that the use of OFN will facilitate the analysis of financial instruments with imprecise estimated values. It is expedient to further development the fuzzy finance theory based on OFN.

In case of analysis of a single financial instrument, we can adopt here the methods recommended in Piasecki (2011a, 2014).

Such broad possibilities encourage further research into the application of ordered fuzzy numbers in the theory and practice of quantified finance.

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