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THE TREYNOR'S RATIO FOR DISCOUNT FACTOR GIVEN AS ORDERED TRAPEZOIDAL FUZZY NUMBER

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Abstract: *In the article, the imprecise present value is evaluated by means of a trapezoidal ordered fuzzy number. Then expected discount factor is a trapezoidal ordered fuzzy number too. The imprecise value of this factor may be used as a decision premise in creating new investment strategies. Considered strategies are built based on a comparison of a fuzzy profit index and the value limit. This way we obtain imprecise investment recommendation given as s fuzzy subset of rating scale. Financial equilibrium criteria result from a special case of this comparison. Further in the paper, the Treynor's Ratio criterion is generalized for the case when expected discount factor is given as ordered trapezoidal fuzzy number. Obtained results show that proposed theory can be used in investment applications.*

Key words: *imprecision, uncertainty, ordered fuzzy number, expected discount factor, investment recommendation*

JEL codes: *C44, C02, G10*

1. Introduction

Present value (PV) is defined in (Piasecki, 2012) as a present equivalent of a payment available in a given time in the future. PV of future cash flows is widely accepted to be an approximate value, with fuzzy numbers being one of the main tools of its modelling. If PV is evaluated by fuzzy number, then the expected return rate is a fuzzy subset in the real line. This result is a theoretical foundation for investment strategies presented in (Piasecki, 2014, 2016).

Ordered fuzzy numbers (OFN) are defined in intuitive way by Kosiński et al (2002) which in this way were going to introduce a fuzzy number supplemented by orientation. Kosiński (2006) has shown that there exist such OFN's which are not fuzzy numbers. For this reason, the original Kosiński's theory was revised in (Piasecki, 2018a).

The main goal of this paper is extension of mentioned above investment strategies for the case when PV of considered security is evaluated by OFN. Moreover, in (Piasecki, Siwek, 2018) it is shown that the fuzzy expected discount factor is a better tool for appraising considered securities than the fuzzy expected return rate. Therefore, we will use expected discount factor premise of decision making. All obtained results will be applied for extension the Treynor's Ratio criterion to the case of PV evaluated by means of OFN.

The paper is organized as follows. Section 2 stands as a theoretical background and outlines fuzzy OFNs and their basic properties. Some method for ordering OFNs is presented in Section 3. In Section 4 PV is imprecisely evaluated by trapezoidal OFNs. The fuzzy expected discount factor is then determined in the Section 5. Next five sections contain Authors' original contribution. Section 6 shows the fuzzy expected discount factor as a primary premise for investment recommendation. It is used to determine the fuzzy expected return rate, which is a secondary premise for investment decision-making. An upgraded model for investment recommendations dependent on expected return system is described in Section 7. Section 8 considers investment recommendations dependent on fuzzy expected return. The Treynor's Ratio Criterion is extended in Section 9. Section 10 concludes the article, summarizes the main findings and proposes some future research directions.

2. Elements of Ordered Fuzzy Numbers

By $\mathcal{F}(\mathbb{X})$ we denote the family of all fuzzy subsets of an arbitrary space $\mathbb{X}$. An imprecise number is a family of values in which each considered value belongs to it in a varying degree. A commonly accepted model of an imprecise number is the fuzzy number (FN), defined as a fuzzy subset of the real line $\mathbb{R}$. The most general definition of FN was given by Dubois and Prade (1978).

Ordered fuzzy numbers (OFN) were intuitively introduced by Kosiński et al (2002) as an extension of the concept of FN. A significant drawback of Kosiński's theory is that there exist such OFNs which are not FN (Kosiński, 2006). The intuitive Kosiński's approach to the notion of OFN is very useful. The OFNs' usefulness follows from the fact that an OFN is defined as FN supplemented by negative orientation or positive one. Negative orientation means the order from bigger numbers to smaller ones. Positive orientation means the order from smaller numbers to bigger ones. The FN orientation is interpreted as prediction of future FN changes. The Kosiński's theory was revised by Piasecki (2018a). OFNs are generally defined in following way:

Definition 1: For any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$ ordered fuzzy number (OFN) $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is defined as the pair of fuzzy number determined by its membership function $\mu_{\mathcal{L}}(\cdot\,|a, b, c, d, S_L, E_L) \in [0; 1]^{\mathbb{R}}$ given by the identity

$$\mu_{\mathcal{L}}(x) = \mu_{\mathcal{FN}}(x|a, b, c, d, S_L, E_L) = \begin{cases} 0, & x \notin [a, d] = [d, a], \\ S_L(x), & x \in [a, b] = [b, a], \\ 1, & x \in [b, c] = [c, b], \\ E_L(x), & x \in [c, d] = [d, c] \end{cases} \tag{1}$$

and orientation $[\![a \rightarrowtail d]\!] = (a, d)$, where the starting-function $S_L \in [0; 1]^{[a,b]}$ and the ending-function $E_L \in [0; 1]^{[c,d]}$ are upper semi-continuous monotonic functions satisfying the conditions

$$S_L(b) = E_L(c) = 1. \tag{2}$$

$$\forall_{x \in]a,d[} \quad \mu_{\mathcal{FN}}(x|a, b, c, d, S_L, E_L) > 0\ .\square \tag{3}$$

Let us note that the identity (1) describes additionally extended notation of numerical intervals, which is used in this work.

The space of all OFN is denoted by the symbol $\mathbb{K}$. The condition $a < d$ fulfilment determines the positive orientation $[\![a \rightarrowtail d]\!]$ of OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$. In this case, the starting-function S_L is non-decreasing and the ending-function E_L is non-increasing. Any positively oriented OFN is interpreted as such imprecise number, which may increase. The space of all positively oriented OFNs we denote by the symbol $\mathbb{K}^+$. The condition $a > d$ fulfilment determines the negative orientation of OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$. In this case, the starting-function S_L is non-increasing and the ending-function E_L is non-decreasing. Negatively oriented OFN is interpreted as such imprecise number, which may decrease. The space of all negatively oriented OFNs we denote by the symbol $\mathbb{K}^-$. For the case $a = d$, OFN

$$[\![a]\!] = \overleftrightarrow{\mathcal{L}}(a, a, a, a, S_L, E_L) \tag{4}$$

represents crisp number $a \in \mathbb{R}$, which is not oriented.

An unary operations on OFNs may be defined in way coherent with the Zadeh's Extension Principle (Zadeh, 1975 a, b c). It means that, for OFN $\overleftrightarrow{\mathcal{L}} \in \mathbb{K}$ represented by its membership function $\mu_{\mathcal{L}} \in [0,1]^{\mathbb{R}}$ and for any function $h: \mathbb{R} \supset \mathbb{A} \to \mathbb{R}$ the value

$$\overleftrightarrow{\mathcal{Z}} = h\left(\overleftrightarrow{\mathcal{L}}\right), \tag{5}$$

is described by its membership function $\mu_{\mathcal{Z}} \in [0;1]^{\mathbb{R}}$ given by the formula:

$$\mu_{\mathcal{Z}}(z) = \sup\{\mu_{\mathcal{L}}(x): z = h(x), x \in \mathbb{A}\}. \tag{6}$$

In special case if the function $h: \mathbb{R} \supset \mathbb{A} \to \mathbb{R}$ is monotonic then we have

$$\mu_{\mathcal{Z}}(x) = \mu_{\mathcal{L}}\left(h^{-1}(x)\right) = \mu_{FN}(x|h(a), h(b), h(c), h(d), S_L \circ h^{-1}, E_L \circ h^{-1}), \tag{7}$$

We will also consider the following special type of OFNs.

Definition 2: If for any nondecreasing sequence $(a, b, c, d) \subset \mathbb{R}$ the starting-function $S_T \in [0;1]^{[a,b]}$ and the ending-function $E_T \in [0;1]^{[c,d]}$ are given by the identities

$$S_T(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a, b[, \\ 1, & x = b, \end{cases} \tag{8}$$

$$E_T(x) = \begin{cases} \frac{x-d}{c-d}, & x \in]c, d], \\ 1, & x = c, \end{cases} \tag{9}$$

then the OFN

$$\overleftrightarrow{\mathcal{T}} = \overleftrightarrow{\mathcal{Tr}}(a, b, c, d) = \overleftrightarrow{\mathcal{L}}(a, b, c, d, S_T, E_T) \tag{10}$$

is called trapezoidal OFN (TrOFN). □

It is obvious that the TrOFN $\overleftrightarrow{\mathcal{T}} = \overleftrightarrow{\mathcal{Tr}}(a, b, c, d)$ is determined by its membership function $\mu_{\mathcal{T}} \in [0,1]^{\mathbb{R}}$ given by the identity

$$\mu_{\mathcal{T}}(x) = \mu_{\mathcal{Tr}}(x|a, b, c, d) = \mu_{\mathcal{FN}}(x|a, b, c, d, S_T, E_T). \tag{11}$$

The space of all TrOFNs we denote by the symbol $\mathbb{K}_{Tr}$.

3. Ordering of Ordered Fuzzy Numbers

Let us consider the pair $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in \mathbb{K} \times \mathbb{K}$ of OFNs. In the space $\mathbb{K}$ we define a relation $\overleftrightarrow{\mathcal{K}} \succcurlyeq \mathcal{L}$ as follows

$$\overleftrightarrow{\mathcal{K}} \succcurlyeq \overleftrightarrow{\mathcal{L}} \Leftrightarrow \text{“}TrOFN\ \overleftrightarrow{\mathcal{K}}\ is\ greater\ or\ equal\ to\ TrOFN\ \overleftrightarrow{\mathcal{L}}.\text{”} \tag{12}$$

This relation is, in fact, a fuzzy preorder $\mathcal{Q} \in \mathcal{F}(\mathbb{K} \times \mathbb{K})$ determined by its membership function $\nu_{\mathcal{Q}} \in [0,1]^{\mathbb{K} \times \mathbb{K}}$ described in detail in (Piasecki, 2018b). From the point view of multivalued logic, the value $\nu_{\mathcal{Q}}\left(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}\right)$ may be interpreted as a truth value of the sentence (12). Let $\overleftrightarrow{\mathcal{K}} \in \mathbb{K}$

be represented by its membership function $\mu_{\mathcal{FN}}(\cdot\,|a,b,c,d,S_K,E_K)$. Due the results obtained in , for any pair $(\overleftrightarrow{\mathcal{K}},h)\in\mathbb{K}\times\mathbb{R}$ we have:

– for $\overleftrightarrow{\mathcal{K}}\in\mathbb{K}^+$

$$\nu_Q\left(\overleftrightarrow{\mathcal{K}},[\![h]\!]\right)=\sup\{\mu_{\mathcal{K}}(x):x\geq h\}=\begin{cases}0, & h>a,\\ S_K(h), & a\geq h>b,\\ 1, & b\geq h,\end{cases} \tag{13}$$

$$\nu_Q\left([\![h]\!],\overleftrightarrow{\mathcal{K}}\right)=\sup\{\mu_{\mathcal{K}}(x):x\leq h\}=\begin{cases}1, & c\leq h,\\ E_K(h), & c>h\geq d,\\ 0, & d>h.\end{cases} \tag{14}$$

– for $\overleftrightarrow{\mathcal{K}}\in\mathbb{K}^-$

$$\nu_Q\left(\overleftrightarrow{\mathcal{K}},[\![h]\!]\right)=\sup\{\mu_{\mathcal{K}}(x):x\geq h\}=\begin{cases}0, & h>d,\\ S_K(h), & d\geq h>c,\\ 1, & c\geq h,\end{cases} \tag{15}$$

$$\nu_Q\left([\![h]\!],\overleftrightarrow{\mathcal{K}}\right)=\sup\{\mu_{\mathcal{K}}(x):x\leq h\}=\begin{cases}1, & b\leq h,\\ E_K(h), & b>h\geq a,\\ 0, & a>h.\end{cases} \tag{16}$$

4. Imprecise Present Value

In the Interest Theory, present value (PV) is defined as discounted cash flow. Generally, PV is defined in (Piasecki, 2012), as the utility of a cash flow, which has a subjective nature due to behavioural influences. Imprecise estimation of PV is a result of the subjective approach to security valuation (Barberis et al, 1998). Therefore, PV is widely accepted as an approximate value, with FNs being one of the main tools of its modelling. In this paper, information on imprecise PV is supplemented with a subjective forecast of orientation of the market price trend. Therefore, imprecise PV may be evaluated by OFN (Łyczkowska-Hanćkowiak, Piasecki, 2018a). Any PV evaluated by OFN is called oriented PV. In agree with OFN interpretation, we use here the following rules recording the alleged orientation of the trend:

– The prediction of rise in market price is described as a positive orientation of OFN.

– The prediction of fall in market price is described as the negative orientation of OFN.

Kuchta (2000) has previously proved the sensibility of trapezoidal fuzzy numbers as a fuzzy financial arithmetic tool. Therefore, we will imprecisely evaluate PVs by TrOFN.

Let us consider a fixed security, for which we observe its market price $\check{C}>0$. Then, in agreement with assumptions introduced above, we can imprecisely determine PV as TrOFN

$$\overleftrightarrow{\mathcal{PV}}=\overleftrightarrow{\mathcal{Tr}}\left(V_s,V_f,V_l,V_e\right), \tag{17}$$

where the monotonic sequence $\left(V_s,V_f,\check{C},V_l,V_e\right)$ is characterized as follows:

- $\check{C}$ - market price,
- $[V_s, V_e] \subset \mathbb{R}^+$ is interval of all possible PV' values,
- $[V_f, V_l] \subset [V_s, V_e]$ is interval of all values which do not perceptible differ from market price $\check{C}$.

Then the PV membership function $\mu_{\mathcal{PV}} = \mu_{\mathcal{Tr}}(\cdot\,|V_s, V_f, V_l, V_e) \in [0,1]^{\mathbb{R}}$ is described by (11).

5. Discount Factor of a Security

By a security we understand an authorization to receive a future financial revenue, payable to a certain maturity. The value of this revenue is interpreted as an anticipated future value (FV) of the asset. Yet, in the researched case, we can point out this particular time in the future, in which the considered income value will be already known to the observer.

Let us assume a fixed time horizon $t > 0$ of an investment. Then, a security is determined by two values: anticipated FV V_t and assessed PV V_0. In (Piasecki & Siwek, 2018) it is justified that FV is a random variable. The basic benefit connected with this security is usually characterized by simple return rate r_t given as follows:

$$r_t = \frac{V_t - V_0}{V_0}. \tag{18}$$

where:

- V_t is a FV described by random variable $\tilde{V}_t: \Omega \longrightarrow \mathbb{R}^+$;
- V_0 is a PV assessed precisely or approximately.

The set Ω is a set of elementary states ω of the financial market. In a classical approach to the problem of return rate estimation, PV of a security is identified with observed market price $\check{C}$. Thus, the return rate is a random variable determined by identity

$$r_t(\omega) = \frac{\tilde{V}_t(\omega) - \check{C}}{\check{C}}. \tag{19}$$

In practice of financial markets analysis, the uncertainty risk is usually described by probability distribution of return rate determined by (19). Nowadays, we have an extensive knowledge about this subject. Let us assume that mentioned probability distribution is given by cumulative distribution function $F_r(\cdot\,|\bar{r}): \mathbb{R} \longrightarrow [0;1]$. We assume that the expected value $\bar{r}$ of this distribution exists. Moreover, let us note that we have

$$\tilde{V}_t(\omega) = \check{C} \cdot \left(1 + r_t(\omega)\right). \tag{20}$$

Let us now consider the case when imprecise PV is estimated by TrOFN $\mathcal{PV}$ represented by its membership function $\mu_{\mathcal{PV}} = \mu_{Tr}(\cdot\,|V_s, V_f, V_l, V_e) \in [0,1]^{\mathbb{R}}$ given by identity (11).

According to (7), and (20), a return rate is a random OFN represented by its membership function $\tilde{\rho} \in [0;1]^{\mathbb{R}\times\Omega}$ as

$$\tilde{\rho}(r,\omega) = \mu_{\mathcal{PV}}\left(\frac{V_t(\omega)}{1+r}\right) = \mu_{\mathcal{PV}}\left(\frac{\check{C}\cdot(1+r_t(\omega))}{1+r}\right). \tag{21}$$

Oriented fuzzy ERR $\mathcal{R} \in \mathbb{K}$ is not TrOFN. In (Piasecki, Siwek, 2018) it is shown that the fuzzy expected discount factor (EDF) is a better tool for appraising considered securities than the fuzzy ERR. Therefore, we aim to determine the fuzzy EDF for the considered case. The EDF $\bar{v}$ is calculated using the return rate $\bar{r}$ by the formula

$$\bar{v} = \frac{1}{1+\bar{r}}. \tag{22}$$

Therefore, the function $\delta \in [0;1]^{\mathbb{R}}$, described by

$$\delta(v) = \delta\left(\frac{1}{1+r}\right) = \rho(r) \tag{23}$$

is a membership function of the oriented fuzzy EDF $\overleftrightarrow{\mathcal{D}} \in \mathbb{K}$ calculated using oriented fuzzy ERR $\overleftrightarrow{\mathcal{R}} \in \mathbb{K}$. The identity (7) implies that

$$\delta(v) = \delta\left(\frac{1}{1+r}\right) = \rho\left(\frac{1}{v}-1\right) = \mu_{PV}\left(\frac{\check{C}\cdot v}{\bar{v}}\right) = \mu_{Tr}\left(v \middle| V_s\cdot\frac{\bar{v}}{\check{C}}, V_f\cdot\frac{\bar{v}}{\check{C}}, V_l\cdot\frac{\bar{v}}{\check{C}}, V_e\cdot\frac{\bar{v}}{\check{C}}\right). \tag{24}$$

The formal simplicity of obtained EDF description encourages for its further application as a portfolio analysis tool. It is proved that for any portfolio, its fuzzy EDF is a simple function of the oriented fuzzy EDFs of components. Therefore, in the next Section we will assume that the basic benefit from a financial instrument is characterized by an oriented fuzzy EDF.

6. Premises of Recommendations

Let us consider an arbitrary financial instrument Φ consisting of a single security. Then, in agreement with assumptions introduced in the previous Section, the basic benefit from this financial instrument is characterized by a oriented EDF

$$\overleftrightarrow{\mathcal{D}} = \overleftrightarrow{\mathcal{Tr}}(D_s, D_f, D_l, D_e), \tag{25}$$

where the monotonic sequence $(D_s, D_f, D_l, D_e) \subset \mathbb{R}^+$ is given as follows:

- $[D_s, D_e] \subset \mathbb{R}^+$ is interval of all possible EDF' values,
- $[D_f, D_l] \subset [D_s, D_e]$ is interval of all values which do not perceptible differ from EDF $\bar{v}$.

Then the EDF membership function $\mu_{\mathcal{D}} = \mu_{\mathcal{Tr}}(\cdot\,|D_s, D_f, D_l, D_e) \in [0,1]^{\mathbb{R}}$ is described by (11). Oriented fuzzy EDF $\mathcal{D} \in \mathbb{F}$ will be considered as a primary premise for investment recommendations.

On the other hand, most well-known models of financial mathematics are described using the concept of ERR. Therefore, using (7) and (25), the secondary investment premise will be designated as fuzzy ERR

$$\overleftrightarrow{\mathcal{R}} = \overleftrightarrow{\mathcal{R}}\left(R_s, R_f, R_l, R_e, L_{\mathcal{R}}, R_{\mathcal{R}}\right), \tag{26}$$

where

- for $i \in \{s, f, l, e\}$ we have

$$R_i = \frac{1}{D_i} - 1, \tag{27}$$

- the starting-function $S_{\mathcal{R}}$ is given for $x \in \left[R_s, R_f\right[$ by the identity

$$S_{\mathcal{R}}(x) = S_{\mathcal{D}}\left(\frac{1}{1+x}\right) = \frac{\frac{1}{1+x} - D_s}{D_f - D_s}, \tag{28}$$

- the ending-function $E_{\mathcal{R}}$ is given for $x \in \left]R_l, R_e\right]$ by the identity

$$E_{\mathcal{R}}(x) = E_{\mathcal{D}}\left(\frac{1}{1+x}\right) = \frac{\frac{1}{1+x} - D_e}{D_l - D_e}. \tag{29}$$

Then the ERR membership function $\mu_{\mathcal{R}} = \mu_{\mathcal{FN}}\left(\cdot \left| R_s, R_f, R_l, R_e, S_{\mathcal{R}}, E_{\mathcal{R}}\right.\right) \in [0,1]^{\mathbb{R}}$ is described by (1).

7. Investment Recommendations Dependent on Expected Return

An investment recommendation is a counsel given by an advisor to the investor. For convenience, these recommendations may be expressed by means of standardized advices. In this article, we will consider the following vocabulary:

- *Buy* **-** suggests that evaluated security is significantly undervalued,
- *Accumulate* - suggests that evaluated security is undervalued,
- *Hold* - suggests that evaluated security is fairly valued,
- *Reduce* - suggests that evaluated security is overvalued,
- *Sell* - suggests that evaluated security is significantly overvalued.

Many advisors use different terminology and vocabulary when forming an advice. Yet, we will concentrate on the five-element advisor's vocabulary considered in (Piasecki, 2014). The advices mentioned above form a set

$$\mathbb{A} = \{A^{++}, A^{+}, A^{0}, A^{-}, A^{--}\}, \tag{30}$$

which is called a rating scale, where

- A^{++} denotes the advice *Buy*,
- A^{+} denotes the advice *Accumulate*,

– A^0 denotes the advice *Hold*,
– A^- denotes the advice *Reduce*,
– A^{--} denotes the advice *Sell*.

Let us take into account a fixed financial instrument Φ represented by ERR $\bar{r}$. For such a case, the advisor's counsel depends on the ERR. The quality criterion for the advice can be given as a comparison of values $g(\bar{r})$ and $\hat{G}$ defined as follows:

– $g: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function of substantially justified form,
– $\hat{G}$ denotes the substantially justified limit value.

The function $g: \mathbb{R} \rightarrow \mathbb{R}$ serves as a profit index. By denoting the powerset of the set $\mathbb{A}$ as $\mathcal{P}(\mathbb{A})$ we define an advice choice function $\Lambda: \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{A})$ as follows

$$\begin{cases} A^{++} \in \Lambda(\bar{r}, \check{G}) \Leftrightarrow g(\bar{r}) > \hat{G} \Leftrightarrow \neg\, g(\bar{r}) \leq \hat{G}, \\ A^{+} \in \Lambda(\bar{r}, \check{G}) \Leftrightarrow g(\bar{r}) \geq \hat{G}, \\ A^{0} \in \Lambda(\bar{r}, \check{G}) \Leftrightarrow g(\bar{r}) = \hat{G} \Leftrightarrow g(\bar{r}) \geq \hat{G} \wedge g(\bar{r}) \leq \hat{G}, \\ A^{-} \in \Lambda(\bar{r}, \check{G}) \Leftrightarrow g(\bar{r}) \leq \hat{G}, \\ A^{--} \in \Lambda(\bar{r}, \check{G}) \Leftrightarrow g(\bar{r}) < \hat{G} \Leftrightarrow \neg\, g(\bar{r}) \geq \hat{G}. \end{cases} \quad (31)$$

This way, we assign the advice subset to a security $\check{S}$. Value $\Lambda(\bar{r}, \check{G})$ is called an investment recommendation. This recommendation may be used as a good starting point for the equity portfolio strategies. On the other hand, the weak point of the proposed choice function is the omission of the fundamental analysis results and the impact of behavioural factors. When analyzing the above stated choice function, it is easy to spot the lack of visible boundaries between the advices *Buy* and *Accumulate* as well as *Reduce* and *Sell*. The justification for this distinction lies in the fundamental analysis and behavioral aspects of the investment process.

8. Investment Recommendations Dependent on Fuzzy Expected Return

Let us assume that the basic benefit from a financial instrument Φ is characterized by a given oriented fuzzy ERR $\overleftrightarrow{\mathcal{R}}$ determined in the Section 6. It implies, that advisor's counsels should be dependent on the oriented fuzzy ERR $\overleftarrow{\mathcal{R}}$, which is determined by its membership function $\mu_{\mathcal{R}} = \mu_{\mathcal{FN}}(\cdot\, | R_s, R_f, R_l, R_e, L_{\mathcal{R}}, R_{\mathcal{R}}) \in [0,1]^{\mathbb{R}}$.

In the first step, we extend the profit index domain $\mathbb{R}$ to the space $\mathbb{F}$. According to (7), the profit index $g(\overleftrightarrow{\mathcal{R}})$ is described by its membership function $\gamma \in [0,1]^{\mathbb{R}}$

$$\gamma(x) = \mu_{\mathcal{R}}(g^{-1}(x)). \quad (32)$$

In the second step we extend the advice choice function's domain $\mathbb{R}^2$ to the set $\mathbb{K} \times \mathbb{R}$. The value of the advice choice function $\Lambda(\overleftrightarrow{\mathcal{R}}, \breve{G})$ is described by its membership function $\lambda(\cdot\, | \overleftrightarrow{\mathcal{R}}, \breve{G}) \in [0,1]^{\mathbb{A}}$, which is determined (due to (31)) by:

$$\lambda(A^{++} | \overleftrightarrow{\mathcal{R}}, \breve{G}) = 1 - \nu_{\mathcal{Q}}\left(\llbracket \breve{G} \rrbracket, g(\overleftrightarrow{\mathcal{R}})\right), \tag{33}$$

$$\lambda(A^{+} | \overleftrightarrow{\mathcal{R}}, \breve{G}) = \nu_{\mathcal{Q}}(g(\overleftrightarrow{\mathcal{R}}), \llbracket \breve{G} \rrbracket), \tag{34}$$

$$\lambda(A^{0} | \overleftrightarrow{\mathcal{R}}, \breve{G}) = \min\left\{\nu_{\mathcal{Q}}(g(\overleftrightarrow{\mathcal{R}}), \llbracket \check{G} \rrbracket), \nu_{\mathcal{Q}}\left(\llbracket \check{G} \rrbracket, g(\overleftrightarrow{\mathcal{R}})\right)\right\}, \tag{35}$$

$$\lambda(A^{-} | \overleftrightarrow{\mathcal{R}}, \check{G}) = \nu_{\mathcal{Q}}\left(\llbracket \breve{G} \rrbracket, g(\overleftrightarrow{\mathcal{R}})\right), \tag{36}$$

$$\lambda(A^{--} | \overleftrightarrow{\mathcal{R}}, \check{G}) = 1 - \nu_{\mathcal{Q}}(g(\overleftrightarrow{\mathcal{R}}), \llbracket \breve{G} \rrbracket). \tag{37}$$

This way we define a fuzzy advice choice function $\Lambda: \mathbb{K} \times \mathbb{R} \longrightarrow \mathcal{F}(\mathbb{A})$. The value $\Lambda(\mathcal{R}, \breve{G})$ is called an imprecise investment recommendation. Moreover, the value $\lambda(A | \overleftrightarrow{\mathcal{R}}, \breve{G})$ may be interpreted as a degree in which the advice $A \in \mathbb{A}$ was chosen.

Above, the ERR is replaced by a fuzzy oriented ERR, which also takes into account fundamental behavioural aspects of decision making in finance. However, such an increase in cognitive value has its price, that is - the imprecise formulation of investment recommendations. By taking into account the imprecision of information about a security, we cannot precisely indicate the recommended action out of a set of advices. Each advice is thus recommended to some extent. The investor shifts some of the responsibility to the advisors. For this reason, the investor restricts their choice of investment decisions to advices recommended to the greatest degree. This way, the investor minimizes their individual responsibility for making a financial decision. However, a final investment decision will, in the end, be made by the investor themselves. Guided by their own knowledge and intuition, an investor can choose an advice which is recommended in a lower degree.

However, let us note that:

$$\lambda(A^{++} | \overleftrightarrow{\mathcal{R}}, \breve{G}) = 1 - \lambda(A^{-} | \overleftrightarrow{\mathcal{R}}, \breve{G}), \tag{38}$$

$$\lambda(A^{0} | \overleftrightarrow{\mathcal{R}}, \check{G}) = \min\{\lambda(A^{+} | \overleftrightarrow{\mathcal{R}}, \breve{G}), \lambda(A^{-} | \overleftrightarrow{\mathcal{R}}, \breve{G})\}, \tag{39}$$

$$\lambda(A^{--} | \overleftrightarrow{\mathcal{R}}, \check{G}) = 1 - \lambda(A^{+} | \overleftrightarrow{\mathcal{R}}, \check{G}). \tag{40}$$

This shows that functions values $\lambda(A^{+} | \overleftrightarrow{\mathcal{R}}, \breve{G})$ and $(A^{-} | \overleftrightarrow{\mathcal{R}}, \breve{G})$ are sufficient to determine an imprecise investment recommendation. For this reason, we will use them when creating a specific method of choosing an advice. Let us note that using an imprecisely estimated return allows for determining differences between advices *Buy* and *Accumulate* and between *Reduce* and *Sell*.

In the next two sections, some example of imprecise recommendations methodologies is presented. All methodologies presented below are generalizations of well know classical methodologies to the fuzzy case.

9. The Treynor's Ratio

We will consider fixed financial instrument Φ which is represented by a pair $(\bar{r}, \beta) \in \mathbb{R} \times \mathbb{R}$ of its ERR $\bar{r}$ and the beta coefficient β of the CAPM model assigned to this instrument. On the capital market we can observe a risk-free return r_0 an the expected market return r_M. Additionally, we assume that the security return is positively correlated with the market return. It implies that $\beta \in \mathbb{R}^+$.

Treynor (1965) defines the profit index $g: \mathbb{R} \to \mathbb{R}$ and the limit value $\hat{G}$ as follows

$$g(\bar{r}) = T(\bar{r}|r_0, \beta) = \frac{\bar{r} - r_0}{\beta}, \tag{41}$$

$$\hat{G} = r_M - r_0. \tag{42}$$

The Treynor's profit index estimates the amount of premium per market risk unit. The Treynor's limit value is equal to the premium of the market risk.

If a financial instrument Φ is represented by a pair $(\overleftrightarrow{\mathcal{R}}, \beta)$ then the oriented profit index is given as OFN $T(\overleftrightarrow{\mathcal{R}}|r_0, \beta)$. The identities (7), (26), (27), (28) and (29) imply that

$$T(\overleftrightarrow{\mathcal{R}}|r_0, \beta) = \overleftrightarrow{\mathcal{T}}(T_s, T_f, T_l, T_e, S_{\mathcal{T}}, E_{\mathcal{T}}), \tag{43}$$

where

- for $i \in \{s, f, l, e\}$ we have

$$T_i = \beta \cdot R_i + r_0 = \beta \cdot \frac{1 - D_i}{D_i} + r_0, \tag{44}$$

- the starting-function $L_{\mathcal{T}}$ is given for $x \in [R_s, R_f[$ by the identity

$$S_{\mathcal{T}}(x) = S_{\mathcal{R}}(\beta \cdot x + r_0) = S_{\mathcal{D}}\left(\frac{1}{1 + \beta \cdot x + r_0}\right) = \frac{\frac{1}{1 + \beta \cdot x + r_0} - D_s}{D_f - D_s}, \tag{45}$$

- the ending-function $E_{\mathcal{T}}$ is given for $x \in]R_l, R_e]$ by the identity

$$E_{\mathcal{T}}(x) = E_{\mathcal{R}}(\beta \cdot x + r_0) = E_{\mathcal{D}}\left(\frac{1}{1 + \beta \cdot x + r_0}\right) = \frac{\frac{1}{1 + \beta \cdot x + r_0} - D_e}{D_l - D_e}. \tag{46}$$

Then the ERR membership function $\mu_{\mathcal{T}} = \mu_{FN}(\cdot | T_s, T_f, T_l, T_e, S_{\mathcal{T}}, E_{\mathcal{T}}) \in [0,1]^{\mathbb{R}}$ is described by (1).

In accordance with (13) – (16) and (43) – (46) we have

– if $T(\overleftrightarrow{\mathcal{R}}|r_0, \beta) \in \mathbb{K}^+$ then

$$\lambda(A^+|\overleftrightarrow{\mathcal{R}}, \breve{G}) = \nu_Q(T(\overleftrightarrow{\mathcal{R}}|r_0, \beta), [\![r_M - r_0]\!]) = \begin{cases} 0, & r_M - r_0 > a, \\ S_{\mathcal{T}}(r_M - r_0), & a \geq r_M - r_0 > b, \\ 1, & b \geq r_M - r_0, \end{cases} \tag{47}$$

$$\lambda(A^-|\overleftrightarrow{\mathcal{R}}, \breve{G}) = \nu_Q\left([\![r_M - r_0]\!], T(\overleftrightarrow{\mathcal{R}}|r_0, \beta)\right) = \begin{cases} 1, & c \leq r_M - r_0, \\ E_{\mathcal{T}}(r_M - r_0), & c > r_M - r_0 \geq d, \\ 0, & d > r_M - r_0. \end{cases} \tag{48}$$

– if $T(\overleftrightarrow{\mathcal{R}}|r_0, \beta) \in \mathbb{K}^-$ then

$$\lambda(A^+|\overleftrightarrow{\mathcal{R}}, \breve{G}) = \nu_Q(T(\overleftrightarrow{\mathcal{R}}|r_0, \beta), [\![r_M - r_0]\!]) = \begin{cases} 0, & r_M - r_0 > d, \\ S_{\mathcal{T}}(r_M - r_0), & d \geq r_M - r_0 > c, \\ 1, & c \geq r_M - r_0, \end{cases} \tag{49}$$

$$\lambda(A^-|\overleftrightarrow{\mathcal{R}}, \breve{G}) = \nu_Q\left([\![r_M - r_0]\!], T(\overleftrightarrow{\mathcal{R}}|r_0, \beta)\right) = \begin{cases} 1, & b \leq r_M - r_0, \\ E_{\mathcal{T}}(r_M - r_0), & b > r_M - r_0 \geq a, \\ 0, & a > r_M - r_0. \end{cases} \tag{50}$$

The values $\lambda(A^{++})$, $\lambda(A^0)$, $\lambda(A^{--})$ are determined respectively by relations (38), (39) and (40).

10. Conclusions

Imprecision is relevant in the investment decision-making process. It is proved above, that imprecise present value and random future value may be considered as a sufficient basis for an investment recommendation.

Obtained results may be applied in behavioural finance theory as a normative model for investment's decisions. The results may also provide theoretical foundations for constructing an investment decision support system.

On the other hand, imprecise estimation of an expected discount factor could be a consequence of taking into account the behavioral aspects of an investment process. Thus, we showed that the behavioral premises can influence investment recommendations in a controlled manner.

In this paper, the main achievement lies in proposing a useful methodology for imprecise investments recommendations. Moreover, the paper also offers original generalization of the Treynor's Ratio Criterion to the fuzzy case. In the crisp case, investment recommendation made by the means of Treynor's Ratio is identical with investment recommendation made by the Jensen's Alpha. An interesting test will be to see if this identity also occurs in the case of PV evaluation using OFN.

Obtained results may well be a starting point for a future investigation of the impact of the present value's imprecision and orientation on imprecision of investment recommendation.

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