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TWO-ASSETS PORTFOLIO WITH TRAPEZOIDAL FUZZY PRESENT VALUES

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Abstract: *The main purpose of the following paper is to present characteristics of a two-asset portfolio in case of present values of composing financial instruments being modelled by a trapezoidal fuzzy number. The future value is described as random variable under the Gaussian distribution of probability. The expected discount factor is defined with the use of simple return rate. Obtained model is an extension of the Markowitz theory to fuzzy case. Throughout the analysis a fuzzy expected discount factor and imprecision risk assessments are calculated. Thanks to that, there arises a possibility to describe the influence of portfolio diversification on imprecision risk. Presented theoretical inference and obtained conclusions are supported by numerical example.*

Key words: *two-assets portfolio, present value, trapezoidal fuzzy number, discount factor*

JEL codes: *C44, C02, G10*

1. Introduction

By a financial asset we understand an authorization to receive a future financial revenue, payable to a certain maturity. The value of this revenue is interpreted as anticipated future value (FV) of the asset. According to the uncertainty theory (von Mises, 1962; Kaplan and Barish, 1967), any unknown future state is uncertain. This uncertainty stems from of our lack of knowledge about the future. Yet, in the researched case, we can point out this particular time in the future, in which the considered income value will be already known to the observer. Behind Kolmogorov (1933, 1956), von Mises (1957), Lambalgen (1996), Sadowski (1980), Czerwiński (1960), and Caplan (2001) we will accept that this is a sufficient condition for modelling the uncertainty with probability. All this leads to the conclusion that FV is a random variable.

The main focus of this research is present value (PV), defined as a present equivalent of a payment available in a given time in the future. PV of a future cash flow is widely accepted to be an approximate value, with fuzzy numbers being one of the main tools of its modelling. A detailed description of the evolution of this particular model can be found in Piasecki (2014). Kuchta (2000) has previously proved the sensibility of using triangular or trapezoidal fuzzy numbers as a fuzzy financial arithmetic tool.

In Piasecki and Siwek (2017) the case of a simple two-asset portfolio with triangular fuzzy present value was researched. After Markowitz (1952), this article was assumes that simple return rates are under Gaussian distribution of probability. There it is shown that, for appraising the considered securities, the expected fuzzy discount factor is better tool than expected return rate. Among other things it was proved that in considered case expected fuzzy discount factor is triangular. Regretfully, entropy measure of an arbitrary triangular fuzzy number is constant, which makes it difficult to analyze the impact of the diversification on imprecision of a portfolio assessment. On the other hand, trapezoidal fuzzy numbers do not have this disadvantage. Therefore, in this paper, expected fuzzy discount factor was applied for securities with trapezoidal fuzzy PV. Obtained in this way results are generalization of results obtained in Piasecki and Siwek (2017).

2. Elements of fuzzy number theory

By $\mathcal{F}(\mathbb{R})$ we denote a family of all fuzzy subsets of a real line $\mathbb{R}$. Dubois and Prade (1979) define a fuzzy number as a fuzzy subset $K \in \mathcal{F}(\mathbb{R})$ with bounded support and represented by membership function $\mu_K \in [0;1]^{\mathbb{R}}$ which satisfies conditions:

$$\exists_{x\in\mathbb{R}}\, \mu_K(x) = 1, \tag{1}$$

$$\forall_{(x,y,z)\in\mathbb{R}^3}\colon x \le y \le z \Longrightarrow \mu_K(y) \ge \min\{\mu_K(x), \mu_K(z)\}. \tag{2}$$

The family of all fuzzy number we denote by the symbol $\mathbb{F}$. Arithmetic operations on fuzzy numbers were defined in (Dubois, Prade 1978). According to the Zadeh's Extension Principle (Zadeh 1965), a sum of fuzzy numbers $K, L \in \mathbb{F}$ represented by their corresponding membership functions $\mu_K, \mu_L \in [0;1]^{\mathbb{R}}$ is a fuzzy subset:

$$M = K \oplus L \tag{3}$$

described by its membership function $\mu_M \in [0;1]^{\mathbb{R}}$:

$$\mu_M(z) = \sup\{\mu_K(x) \wedge \mu_L(z - x)\colon x \in \mathbb{R}\}. \tag{4}$$

Analogously, the multiplication of a real number $\gamma \in \mathbb{R} \setminus \{0\}$ and a fuzzy number $K \in \mathbb{F}$ represented by membership function $\mu_K \in [0;1]^{\mathbb{R}}$ is a fuzzy subset:

$$N = \gamma \odot K \tag{5}$$

described by its membership function $\mu_N \in [0;1]^{\mathbb{R}}$:

$$\mu_N(z) = \mu_K\left(\frac{z}{y}\right). \tag{6}$$

Moreover, if $\gamma = 0$, then the multiplication (5) is equal to zero. The class of fuzzy real numbers is closed under the operations (3) and (5).

Fuzzy numbers are widely used as a model for assessment or estimation of a parameter, which is given imprecisely. Following Klir (1993) we understand imprecision as a superposition of ambiguity and indistinctness of information. Ambiguity can be interpreted as a lack of a clear recommendation between one alternative among various others. Indistinctness is understood as a lack of explicit distinction between recommended and not recommended alternatives. An increase in information imprecision makes it less useful and therefore it is logical to consider the problem of imprecision assessment.

We measure the ambiguity of a fuzzy number by applying the measure proposed by Khalili (1979) to the energy measure $d\colon \mathcal{F}(\mathbb{R}) \to \mathbb{R}_0^+$ defined by de Luca and Termini (1979). For an arbitrary fuzzy number $K \in \mathbb{F}$, with membership function $\mu_K \in [0;1]^{\mathbb{R}}$ we have:

$$d(K) = \int_{-\infty}^{+\infty} \mu_K(x)dx. \tag{7}$$

The indistinctness of an arbitrary fuzzy number can be measured by its entropy $e\colon \mathcal{F}(\mathbb{R}) \to \mathbb{R}_0^+$ (de Luca and Termini, 1979) in form given by Kosko (1986). For an arbitrary fuzzy number $K \in \mathbb{F}$ we have:

$$e(K) = \frac{d(K \cap K^C)}{d(K \cup K^C)}. \tag{8}$$

The main focus of this study is a trapezoidal fuzzy number. The fuzzy number $Tr(r,s,t,u)$ is defined for a non-decreasing sequence $\{r,s,t,u\} \subset \mathbb{R}$ by membership function $\mu(\cdot\,|r,s,t,u) \in [0,1]^{\mathbb{R}}$ by formula:

$$\mu(x|r,s,t,u) = \begin{cases} 0 & \text{for} \quad x < r, \\ \dfrac{x-r}{s-r} & \text{for} \quad r \le x < s, \\ 1 & \text{for} \quad s \le x \le t, \\ \dfrac{x-u}{t-u} & \text{for} \quad t < x \le u, \\ 0 & \text{for} \quad x > u, \end{cases} \tag{9}$$

is a trapezoidal fuzzy number. For any arbitrary pair of trapezoidal fuzzy numbers, $Tr(r_1,s_1,t_1,u_1)$ and $Tr(r_2,s_2,t_2,u_2)$ and $a, b \in \mathbb{R}_0^+$ we have:

$$\begin{gathered} Tr(a\cdot r_1 + b\cdot r_2, a\cdot s_1 + b\cdot s_2, a\cdot t_1 + b\cdot t_2, a\cdot u_1 + b\cdot u_2) = \\ = \big(a \odot Tr(r_1,s_1,t_1,u_1)\big) \oplus \big(b \odot Tr(r_2,s_2,t_2,u_2)\big), \end{gathered} \tag{10}$$

$$d\big(Tr(r_1, s_1, t_1, u_1)\big) = \frac{1}{2} \cdot (u_1 + t_1 - r_1 - s_1), \tag{11}$$

$$e\big(Tr(r_1, s_1, t_1, u_1)\big) = \frac{s_1 - r_1 + u_1 - t_1}{-s_1 - 3 \cdot r_1 + 3 \cdot u_1 + t_1}. \tag{12}$$

$$\mu_{Tr}(\gamma \cdot x | r_1, s_1, t_1, u_1) = \mu_{Tr}\left(x \middle| \frac{r_1}{\gamma}, \frac{s_1}{\gamma}, \frac{t_1}{\gamma}, \frac{u_1}{\gamma}\right), \tag{13}$$

where $\gamma \in \mathbb{R}_0^+$.

3. Return rate from a financial asset

All considerations in this and the following chapter will be performed for a fixed time $t > 0$. We will use a simple return rate r_t defined by the equation:

$$r_t = \frac{V_t - V_0}{V_0}, \tag{14}$$

where:

- V_t is a FV described by random variable $\tilde{V}_t : \Omega \to \mathbb{R}$;
- V_0 is a PV assessed precisely or imprecisely.

For any elementary state $\omega \in \Omega$ of financial market, the variable FV is described by the relationship:

$$\tilde{V}_t(\omega) = \check{C} \cdot \big(1 + \tilde{r}_t(\omega)\big), \tag{15}$$

where the simple return rate $\tilde{r}_t : \Omega \to \mathbb{R}$ is determined for the PV equal to the market price $\check{C}$. It is obvious that the return rate $\tilde{r}_t$ is a random variable with probability distribution described by its cumulative function $F_r : \mathbb{R} \longrightarrow [0; 1]$. After Markowitz (1952) we assume that the return rate $\tilde{r}_t$ has a Gaussian distribution $N(\bar{r}, \sigma)$ of probability. In this paper we additionally assume that the PV is imprecisely estimated by trapezoidal fuzzy number:

$$\mathcal{PV} = DTr\big(\check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\big) \tag{16}$$

with membership function $\mu_{PV}\big(\cdot \,\big|\check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\big) \in [0; 1]^{\mathbb{R}}$. The PV's parameters are interpreted as follows:

- $\check{C}$ is the market price,
- $\check{C}_{min} \in \big]0, \check{C}\big]$ is the maximal lower bound of PV,
- $\check{C}_{max}$ $\big[\check{C}, +\infty\big[$ is the minimal upper bound of PV,
- $\check{C}_* \in \big[\check{C}_{min}, \check{C}\big]$ is the minimal upper assessment of prices visibly lower than the market price $\check{C}$,
- $\check{C}^* \in \big[\check{C}, \check{C}_{max}\big]$ is the maximal lower assessment of prices visibly higher than the market price $\check{C}$.

Method of determining parameters $\check{C}_{min}, \check{C}_{max}$ is given in Piasecki and Siwek (2015).

According to the Zadeh's Extension Principle, the simple return rate calculated for the PV assessed by this method is a fuzzy probabilistic set represented by its membership function $\tilde{\rho} \in [0; 1]^{\mathbb{R} \times \Omega}$ given by:

$$\tilde{\rho}(r, \omega) = \sup\left\{\mu_{PV}\big(x \big| \check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\big) : x = \frac{\tilde{V}_t(\omega)}{1+r}, x \in \mathbb{R}\right\} =$$
$$\mu_{PV}\left(\frac{\tilde{V}_t(\omega)}{1+r} \middle| \check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\right) = \mu_{PV}\left(\check{C} \cdot \frac{1+\tilde{r}_t(\omega)}{1+r} \middle| \check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\right), \tag{17}$$

Then the membership function $\rho \in [0; 1]^{\mathbb{R}}$ of expected return rate is calculated in following way:

$$\rho(r) = \int_{-\infty}^{+\infty} \mu_{PV}\left(\check{C} \cdot \frac{1+y}{1+r} \middle| \check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\right) d\, F_r(y) = \mu_{PV}\left(\check{C} \cdot \frac{1+\bar{r}}{1+r} \middle| \check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\right). \tag{18}$$

It is very easy to see that expected return rate obtained above is not trapezoidal fuzzy number. Therefore, we shall consider expected discount factor $\bar{v}$ defined by identity:

$$\bar{v} = \frac{1}{1+\bar{r}}. \tag{19}$$

In line with (19), the membership function $\delta \in [0,1]^{\mathbb{R}}$ of an discount factor is given by the identity:

$$\delta(v) = \delta\left(\frac{1}{1+r}\right) = \rho(r) = \rho\left(\frac{1}{v} - 1\right). \tag{20}$$

Combining both (13), (18), and (20) we get:

$$\delta(v) = \mu_{PV}\left(\check{C} \cdot \frac{1+\bar{r}}{1+\frac{1}{v}-1} \middle| \check{C}_{min}, \check{C}_{*}, \check{C}^{*}, \check{C}_{max}\right) = \mu_{PV}\left(\frac{\check{C}}{\bar{v}} \cdot v \middle| \check{C}_{min}, \check{C}_{*}, \check{C}^{*}, \check{C}_{max}\right) =$$

$$= \mu_{PV}\left(v \middle| \frac{\check{C}_{min}}{\check{C}} \cdot \bar{v}, \frac{\check{C}_{*}}{\check{C}} \cdot \bar{v}, \frac{\check{C}^{*}}{\check{C}} \cdot \bar{v}, \frac{\check{C}_{max}}{\check{C}} \cdot \bar{v}\right). \tag{21}$$

where $\bar{v}$ is a discount factor appointed using the expected return rate $\bar{r}$. It is easy to see that the discounting factor $\mathcal{V} \in \mathbb{F}$ appointed above is a trapezoidal fuzzy number given by the formula:

$$\mathcal{V} = Tr\left(\frac{\check{C}_{min}}{\check{C}} \cdot \bar{v}, \frac{\check{C}_{*}}{\check{C}} \cdot \bar{v}, \frac{\check{C}^{*}}{\check{C}} \cdot \bar{v}, \frac{\check{C}_{max}}{\check{C}} \cdot \bar{v}\right). \tag{22}$$

The increase in the ambiguity of an expected discount factor $\mathcal{V} \in \mathbb{F}$ leads to an increase in the number of alternative investment recommendations. It implies an increase in the risk of choosing such a financial decision, which will be burdened *ex post* by the lost profit. This kind of risk is called an ambiguity risk. The ambiguity risk burdening the expected discount factor $\mathcal{V}$ is evaluated by the energy measure $d(\mathcal{V})$. According to (11), it equals:

$$d(\mathcal{V}) = \frac{\bar{v}}{2 \cdot \check{C}} \cdot \left(\check{C}_{max} + \check{C}^{*} - \check{C}_{min} - \check{C}_{*}\right). \tag{23}$$

An increase in the indistinctness of a factor $\mathcal{V}$ means that the boundaries distinguishing recommended decision alternatives are getting blurred. This results in an increase in the risk of choosing a not recommended decision. This kind of risk is called an indistinctness risk. The indistinctness risk burdening the expected discount factor $\mathcal{V} \in \mathbb{F}$ is evaluated by the entropy measure $e(\mathcal{V})$. According to (12), it equals:

$$e(\mathcal{V}) = \frac{\check{C}_{*} - \check{C}_{min} + \check{C}_{max} - \check{C}^{*}}{-\check{C}_{min} - 3 \cdot \check{C}_{*} + 3 \cdot \check{C}_{max} - \check{C}^{*}}. \tag{24}$$

The ambiguity risk and vagueness risk combined together will refer to as an imprecision risk.

In each of the considered cases, the return rate is a function of FV, which is uncertain by its nature, as mentioned in the Introduction. This uncertainty stems from an investor's lack of knowledge about future state of affairs. This lack of knowledge implies that no investor is sure of their future profits or losses. An increase of uncertainty can result in higher risk of choosing a wrong financial decision. This type of risk is called an uncertainty risk. The properties of such risk are discussed in a rich body of literature. In this paper, we evaluate the uncertainty risk using the variance σ^2 of the return rate.

The formal simplicity of obtained description of an expected discount factor encourages for its further application as a portfolio analysis tool. The maximization criterion of expected return rate can then be substituted by minimization criterion of the expected discount factor. In case of non-fuzzy values of both parameters, those criteria are equivalent.

An increase in ambiguity of expected discount factor $\mathcal{V} \in \mathbb{F}$ suggests a higher number of alternative recommendations to choose from. This may result in making a decision, which will be *ex post* associated with profit lower than maximal, so with lost chances. This kind of risk is called an ambiguity risk. The ambiguity risk implied by $\mathcal{V}$ is measured by energy measure $d(\mathcal{V})$.

An increase in the indistinctness of $\mathcal{V}$, on the other hand, suggests that the differences between recommended and unrecommended decision alternatives are harder to differentiate. This leads to an increase in the indistinctness risk, which is in a risk of choosing a not recommended option. The indistinctness risk of an expected discount factor $\mathcal{V}$ is measured by entropy measure $e(\mathcal{V})$. Imprecision risk consists of both ambiguity and indistinctness risk combined.

From (14) we have, that the return rate is a function of the future value of an asset, which is uncertain, since we don't know the future state of the world. Because of this, the investor is not sure, whether they will gain or lose from the decision they made. With the increase in uncertainty, the risk of making a wrong decision is higher. Here, uncertainty risk of a return rate will be measured by its variance σ^2.

4. Two-asset portfolio

By a financial portfolio we will understand an arbitrary, finite set of financial assets. Each of this assets is characterized by its assessed PV and anticipated return rate.

Let us consider the case of a two-asset portfolio π, consisting of financial assets Y_1 and Y_2. The PV of assets Y_i $(i = 1; 2)$ is estimated by fuzzy trapezoidal number $Tr\left(\check{C}_{min}^{(i)}; \check{C}_{*}^{(i)}; \check{C}^{*(i)}; \check{C}_{max}^{(i)}\right)$ where parameters are given as follows:

- $\check{C}^{(i)}$ is the market price,
- $\check{C}_{min}^{(i)} \in \left]0; \check{C}^{(i)}\right]$ is the maximal lower bound of PV,

- $\check{C}_{max}^{(i)} \in \left[\check{C}^{(i)};\ +\infty\right[$ is the minimal upper bound of PV,
- $\check{C}_{*}^{(i)} \in \left[\check{C}_{min}^{(i)};\ \check{C}^{(i)}\right]$ is the minimal upper assessment of prices visibly lower than the market price $\check{C}^{(i)}$,
- $\check{C}^{*(i)} \in \left[\check{C}^{(i)}; \check{C}_{max}^{(i)}\right]$ is the maximal lower assessment of prices visibly higher than the market price $\check{C}^{(i)}$.

We assume that for each security $Y_i (i = 1; 2)$ we know the simple return rate $\tilde{r}_t^i : \Omega \rightarrow \mathbb{R}$ appointed by (14) for the PV equal to the market price $\check{C}^{(i)}$. After Markowitz (1952) we assume that the two-dimensional variable $(\tilde{r}_t^1 {}_, \tilde{r}_t^2)^T$ has a cumulative Gaussian distribution $N((\bar{r}_1, \bar{r}_2)^{\mathrm{T}}, \mathbf{\Sigma})$, with a covariance matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & cov_{12} \\ cov_{12} & \sigma_2^2 \end{pmatrix}. \tag{25}$$

We appoint an expected discount factor of security Y_i:

$$\mathcal{V}^{(i)} = Tr\left(\check{C}_{min}^{(i)} \cdot \frac{\bar{v}_i}{\check{C}^{(i)}}; \check{C}_{*}^{(i)} \cdot \frac{\bar{v}_i}{\check{C}^{(i)}}; \check{C}^{*(i)} \cdot \frac{\bar{v}_i}{\check{C}^{(i)}}; \check{C}_{max}^{(i)} \cdot \frac{\bar{v}_i}{\check{C}^{(i)}}\right), \tag{26}$$

where $\bar{v}_i$ is an expected discount factor appointed using the expected return rate $\bar{r}_i$. According to (23), the energy measure of $\mathcal{V}^{(i)}$ is given by:

$$d\left(\mathcal{V}^{(i)}\right) = \frac{\bar{v}_\iota}{2 \cdot \check{C}^i} \cdot \left(\check{C}^{*(i)} + \check{C}_{max}^{(i)} - \check{C}_{min}^{(i)} - \check{C}_{*}^{(i)}\right), \tag{27}$$

and from (24), the entropy measure of a discounting factor can be calculated as:

$$e\left(\ \mathcal{V}^{(i)}\right) = \frac{\check{C}^{*(i)} - \check{C}_{min}{}^{i} \check{C}^{i}{}_{min} + \check{C}_{max}^{(i)} - \check{C}_{*}^{(i)}}{-\check{C}_{*}^{(i)} - 3 \cdot \check{C}_{min}^{(i)} + 3 \cdot \check{C}_{max}^{(i)} + \check{C}^{*(i)}}. \tag{28}$$

We have that the market value $\check{C}^{\ (\pi)}$ of a portfolio π is equal to:

$$\check{C}^{(\pi)} = \check{C}^{(1)} + \check{C}^{(2)}. \tag{29}$$

Share p_i of an instrument Y_i in the portfolio π is given by:

$$p_i = \frac{\check{C}^{(i)}}{\check{C}^{(\pi)}}. \tag{30}$$

Then expected portfolio return rate $\bar{r}$ equals:

$$\bar{r} = p_1 \cdot \bar{r}_1 + p_2 \cdot \bar{r}_2. \tag{31}$$

As for the present value of the portfolio, according to (10) it is also a trapezoidal fuzzy number :

$$PV^{(\pi)} = Tr\left(\check{C}_{min}^{(1)} + \check{C}_{min}^{(2)}; \check{C}_{*}^{(1)} + \check{C}_{*}^{(2)}; \check{C}^{*(1)} + \check{C}^{*(2)}; \check{C}_{max}^{(1)} + \check{C}_{max}^{(2)}\right) = Tr\left(\check{C}_{min}^{(\pi)}; \check{C}_{*}^{(\pi)}; \check{C}^{*(\pi)}; \check{C}_{max}^{(\pi)}\right). \tag{32}$$

By (23), one can calculate the fuzzy expected discount factor of the portfolio π:

$$\mathcal{V}^{(\pi)} = T\left(\check{C}_{min}^{(\pi)} \cdot \frac{\bar{v}}{\check{C}^{(\pi)}}, \check{C}_{*}^{(\pi)} \cdot \frac{\bar{v}}{\check{C}^{(\pi)}}, \check{C}^{*(\pi)} \cdot \frac{\bar{v}}{\check{C}^{(\pi)}}, \check{C}_{max}^{(\pi)} \cdot \frac{\bar{v}}{\check{C}^{(\pi)}}\right), \tag{33}$$

where $\bar{v}$ is a discounting factor calculated for expected return rate $\bar{r}$. We have:

$$\frac{1}{\bar{v}} = \frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}. \tag{34}$$

from which we obtain:

$$\bar{v} = \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} = \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \cdot (p_1 + p_2) = \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \cdot \left(\frac{p_1}{\bar{v}_1} \cdot \bar{v}_1 + \frac{p_2}{\bar{v}_2} \cdot \bar{v}_2\right), \tag{35}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \check{C}_{min}^{(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \left(\check{C}_{min}^{(1)} + \check{C}_{min}^{(2)}\right) = \bar{v} \cdot \left(p_1 \cdot \frac{\check{C}_{min}^{(1)}}{\check{C}^{(1)}} + p_2 \cdot \frac{\check{C}_{min}^{(2)}}{\check{C}^{(2)}}\right) =$$

$$\left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \cdot \left(\frac{p_1}{\bar{v}_1} \cdot \left(\bar{v}_1 \cdot \frac{\check{C}_{min}^{(1)}}{\check{C}^{(1)}}\right) + \frac{p_2}{\bar{v}_2} \cdot \left(\bar{v}_2 \cdot \frac{\check{C}_{min}^{(2)}}{\check{C}^{(2)}}\right)\right), \tag{36}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \check{C}_{*}^{(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \left(\check{C}_{*}^{(1)} + \check{C}_{*}^{(2)}\right) = \bar{v} \cdot \left(p_1 \cdot \frac{\check{C}_{*}^{(1)}}{\check{C}^{(1)}} + p_2 \cdot \frac{\check{C}_{*}^{(2)}}{\check{C}^{(2)}}\right) =$$

$$= \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \cdot \left(\frac{p_1}{\bar{v}_1} \cdot \left(\bar{v}_1 \cdot \frac{\check{C}_*^{(1)}}{\check{C}^{(1)}}\right) + \frac{p_2}{\bar{v}_2} \cdot \left(\bar{v}_2 \cdot \frac{\check{C}_*^{(2)}}{\check{C}^{(2)}}\right)\right), \tag{37}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \check{C}^{*(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \left(\check{C}^{*(1)} + \check{C}^{*(2)}\right) = \bar{v} \cdot \left(p_1 \cdot \frac{\check{C}^{*(1)}}{\check{C}^{(1)}} + p_2 \cdot \frac{\check{C}^{*(2)}}{\check{C}^{(2)}}\right) =$$

$$= \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \cdot \left(\frac{p_1}{\bar{v}_1} \cdot \left(\bar{v}_1 \cdot \frac{\check{C}^{*(1)}}{\check{C}^{(1)}}\right) + \frac{p_2}{\bar{v}_2} \cdot \left(\bar{v}_2 \cdot \frac{\check{C}^{*(2)}}{\check{C}^{(2)}}\right)\right), \tag{38}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \check{C}_{max}^{(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}} \cdot \left(\check{C}_{max}^{(1)} + \check{C}_{max}^{(2)}\right) = \bar{v} \cdot \left(p_1 \cdot \frac{\check{C}_{max}^{(1)}}{\check{C}^{(1)}} + p_2 \cdot \frac{\check{C}_{max}^{(2)}}{\check{C}^{(2)}}\right) =$$

$$= \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \cdot \left(\frac{p_1}{\bar{v}_1} \cdot \left(\bar{v}_1 \cdot \frac{\check{C}_{max}^{(1)}}{\check{C}^{(1)}}\right) + \frac{p_2}{\bar{v}_2} \cdot \left(\bar{v}_2 \cdot \frac{\check{C}_{max}^{(2)}}{\check{C}^{(2)}}\right)\right). \tag{39}$$

From the formulas given above, we can rewrite the fuzzy discount factor as:

$$\mathcal{V}^{(\pi)} = \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \odot \left(\left(\frac{p_1}{\bar{v}_1} \odot \mathcal{V}^{(1)}\right) \oplus \left(\frac{p_2}{\bar{v}_2} \odot \mathcal{V}^{(2)}\right)\right). \tag{40}$$

By (11) and (40), we obtain that the energy measure of an expected discounting factor $\mathcal{V}^{(\pi)} \in \mathbb{F}$ is a linear combination of energy measures calculated for each of component assets:

$$d\left(\mathcal{V}^{(\pi)}\right) = \left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \cdot \left(\frac{p_1}{\bar{v}_1} \cdot d\left(\mathcal{V}^{(1)}\right) + \frac{p_2}{\bar{v}_2} \cdot d\left(\mathcal{V}^{(2)}\right)\right). \tag{41}$$

The relation stated above suggests, that the energy of a fuzzy expected discount factor of a portfolio π is, in fact, a linear combination of weighted energies of those factors calculated for its components. The weights calculated for the assets Y_i increase with their shares in the portfolio and, respectively, decrease with the value of their discount factor $\bar{v}_i$. This fact leads to a conclusion, that when trying to minimize the ambiguity risk of a portfolio, one should focus on minimizing the ambiguity of component assets, which are characterized by the highest expected return rates. On the other hand, the shares of an asset in the whole portfolio are, according to the theory, appointed *post factum,* by gathering available information on said assets. Condition (41) shows that, in the researched case, the portfolio diversification only "averages" the risk of ambiguity.

According to (12), the entropy measure of expected discount factor is equal to:

$$e\left(\mathcal{V}^{(\pi)}\right) = \frac{\check{C}_*^{(\pi)} - \check{C}_{min}^{(\pi)} + \check{C}_{max}^{(\pi)} - \check{C}^{*(\pi)}}{-\check{C}_*^{(\pi)} - 3 \cdot \check{C}_{min}^{(\pi)} + 3 \cdot \check{C}_{max}^{(\pi)} + \check{C}^{*(\pi)}}. \tag{42}$$

The variance of a portfolio return rate is calculated by:

$$\sigma^2 = p_1^2 \cdot \sigma_1^2 + 2 \cdot p_1 \cdot p_2 \cdot cov_{12} + p_2^2 \cdot \sigma_2^2. \tag{43}$$

By constructing a portfolio which minimizes the variance Markowitz proved that portfolio diversification can "minimize" the uncertainty risk.

5. Case study

The portfolio π consists of financial assets Y_1 and Y_2. Anticipated vector $(\tilde{r}_t^1, \tilde{r}_t^2)^T$ of their simple return rates has two-dimensional Gaussian distribution:

$$N\left((0.25, 0.5)^T, \begin{bmatrix} 0.5 & -0.1 \\ -0.1 & 0.4 \end{bmatrix}\right).$$

For the asset Y_1 with market price $\check{C}^{(1)} = 24$, its PV is estimated by a trapezoidal fuzzy number $Tr(18, 23, 25, 37)$. Then we appoint by means of (26) a fuzzy expected discount factor $\mathcal{V}^{(1)} \in \mathbb{F}$. We have:

$$\mathcal{V}^{(1)} = Tr\left(18 \cdot \frac{0.8}{24}, 23 \cdot \frac{0.8}{24}, 25 \cdot \frac{0.8}{24}, 37 \cdot \frac{0.8}{24}\right) = Tr(0.6, 0.77, 0.83, 1.23).$$

According to (23), its energy measure equals:

$$d\left(\mathcal{V}^{(1)}\right) = \frac{0{,}8}{2 \cdot 24} \cdot (37 - 18 + 25 - 23) = 0.35,$$

and from (24), the entropy measure has the value of

$$e(\mathcal{V}^{(1)}) = \frac{23-18+37-25}{-23-3\cdot18+3\cdot37+25} = 0.29.$$

For the second asset Y_2 its expected discount factor $\mathcal{V}^{(2)} \in$ equals:

$$\mathcal{V}^{(2)} = Tr\left(66\cdot\frac{0,67}{69}; 67\cdot\frac{0,67}{69}; 70\cdot\frac{0,67}{69}; 75\cdot\frac{0,67}{69}\right) = Tr(0.64; 0.65; 0.68; 0.73).$$

Also, its energy measure equals:

$$d(\mathcal{V}^{(2)}) = \frac{0,67}{2\cdot69}\cdot(75-66+70-67) = 0.06,$$

and entropy measure has the value of:

$$e(\mathcal{V}^{(2)}) = \frac{67-66+75-70}{-67-3\cdot66+3\cdot75+70} = 0.2.$$

The market price of portfolio π is equal to:

$$\check{C}^{(\pi)} = 24 + 69 = 93.$$

Corresponding to (30), shares p_1 and p_2 of Y_1 and Y_2 in the portfolio π are equal:

$$p_1 = \frac{24}{93},\ \ p_2 = \frac{69}{93}.$$

Using (40), we appoint the fuzzy expected discount factor $\mathcal{V}^{(\pi)} \in \mathbb{F}$ of the portfolio π as trapezoidal fuzzy number of the form:

$$\mathcal{V}^{(\pi)} = \left(\left(\left(\frac{\frac{24}{93}}{0.8}+\frac{\frac{69}{93}}{0.67}\right)^{-1}\cdot\frac{\frac{24}{93}}{0.8}\right)\odot\mathcal{V}^{(1)}\right)\oplus\left(\left(\left(\frac{\frac{24}{93}}{0.8}+\frac{\frac{69}{93}}{0.67}\right)^{-1}\cdot\frac{\frac{69}{93}}{0.67}\right)\odot\mathcal{V}^{(2)}\right) =$$

$$= \left(0.2256\odot\mathcal{V}^{(1)}\right)\oplus\left(0.7744\odot\mathcal{V}^{(2)}\right) = Tr(0.63, 0.68, 0.71, 0.84).$$

Its energy measure calculated by (41) equals:

$$d(\mathcal{V}^{(\pi)}) = 0.2256\cdot0.35 + 0.7744\cdot0.06 = 0.13.$$

Entropy measure can be calculated by (42):

$$e(\mathcal{V}^{(\pi)}) = e\left(Tr(0.63, 0.68, 0.71, 0.84)\right) = \frac{0.68-0.63+0.84-0.71}{-0.68-3\cdot0.63+3\cdot0.84+0.71} = 0,27.$$

Let us note that we have:

$$\left(\frac{p_1}{\bar{v}_1}+\frac{p_2}{\bar{v}_2}\right)^{-1}\cdot\left(\frac{p_1}{\bar{v}_1}\cdot e(\mathcal{V}^{(1)})+\frac{p_2}{\bar{v}_2}\cdot e(\mathcal{V}^{(2)})\right) =$$

$$= 0.2256\cdot0.29 + 0.7744\cdot0.2 = 0.2203 \neq 0,27 = e(\mathcal{V}^{(\pi)}).$$

It implies that the portfolio entropy measure $e(\mathcal{V}^{(\pi)})$ cannot be calculated by analogous way as the portfolio energy measure $e(\mathcal{V}^{(\pi)})$ by the linear combination (41).

We obtain following relations between the energy measure and entropy measure appointed for fuzzy expected discount factors of portfolio and its components:

$$d(\mathcal{V}^{(1)}) > d(\mathcal{V}^{(\pi)}) > d(\mathcal{V}^{(2)}),$$

$$e(\mathcal{V}^{(1)}) > e(\mathcal{V}^{(\pi)}) > e(\mathcal{V}^{(2)}).$$

These inequalities show that the portfolio diversification can average the imprecision risk. Moreover, using (43) we calculate the variance of a return rate from portfolio:

$$\sigma^2 = 0.2175.$$

By increasing the number of assets in the portfolio, we can lower the variance (which with number of assets going to infinity approaches its limit). This means that creating a multi asset portfolio π results in minimizing the uncertainty risk.

Let us consider now any portfolio π consisting of financial assets Y_1 and Y_2. The contribution of the instrument Y_i in the portfolio π is equal to p_i. Then, according to (40), the expected discount factor $\mathcal{V}^{(\pi)} \in \mathbb{F}$ of the portfolio π can be calculated in the following way:

$$D^{(\pi)} = \left(\frac{p_1}{0.8} + \frac{p_2}{0.67}\right)^{-1} \odot \left(\left(\frac{p_1}{0.8} \odot Tr(0.6, 0.77, 0.83, 1.23)\right) \oplus \left(\frac{p_2}{0.67} \odot Tr(0.64; 0.65; 0.68; 0.73)\right)\right) =$$

$$= \frac{0.67 \cdot p_1 \odot Tr(0.6,0.77,0.83,1.23) \oplus 0.8 \cdot p_2 \odot Tr(0.64;0.65;0.68;0.73)}{0.667 \cdot p_1 + 0.8 \cdot p_2} =$$

$$= \frac{p_1 \odot Tr(0.402,0.5159,0.5561,0.8241) \oplus p_2 \odot Tr(0.512,0.52,0.544,0.584\,)}{0.67 \cdot p_1 + 0.8 \cdot p_2}.$$

We see that the expected fuzzy discount factor of portfolio can be expressed as a combination of securities contributions and their expected fuzzy discount factors. In an analogous way the ambiguity risk may be evaluated because of the energy measure for this factor by (41) is given as follows:

$$d\left(\mathcal{V}^{(\pi)}\right) = \left(\frac{p_1}{0.8} + \frac{p_2}{0.67}\right)^{-1} \cdot \left(\frac{p_1}{0.8} \cdot 0.35 + \frac{p_2}{0.67} \cdot 0.06\right) = \frac{0.67 \cdot p_1 \cdot 0.35 + 0.8 \cdot p_2 \cdot 0.06}{0.67 \cdot p_1 + 0.8 \cdot p_2} = \frac{0.2345 \cdot p_1 + 0.048 \cdot p_2}{0.67 \cdot p_1 + 0.8 \cdot p_2}.$$

Above we have shown that entropy measure $e\left(\mathcal{V}^{(\pi)}\right)$ cannot be expressed in analogous way. The last two equations can be applied to the mathematical programing task dedicated to portfolio optimization.

6. Conclusions

The main purpose of this article was to analyse the possibility of managing the risk burdening a two-asset portfolio, built with use of imprecise information stemming from present value of component assets. The imprecise present values were modelled with by trapezoidal fuzzy numbers. For this assumption we have reached the following conclusions:

- The portfolio diversification can lower uncertainty risk.
- The portfolio diversification averages ambiguity risk.
- The portfolio diversification can to average indistinctness risk.

Obtained results suggest, on one hand, that the portfolio diversification does not help in lowering the imprecision risk, but on the other hand, it also does not increase it. Thus, research suggests that there exist portfolios, which imprecision risk will not be minimized with portfolio diversification, and thus it is vital to create a new risk minimization problem, including all of the risk types.

The results obtained above encourage for their broader analysis. Further research can focus on generalizing the representation of the present value to an arbitrary fuzzy number. By using the mathematical induction, all results obtained this way can be generalized to the case of a multi assets portfolio.

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