THE RELATIONS "LESS OR EQUAL" AND "LESS THAN" FOR ORDERED FUZZY NUMBERS

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Abstract: *The ordered fuzzy number is defined as fuzzy number supplemented with orientation of number. Any positively oriented ordered fuzzy number is interpreted as an imprecise real number which may to increase. Any negatively oriented ordered fuzzy number is interpreted as an imprecise real number which may to decrease. The main goal of this paper is to introduce the preorder "less or equal" and the strict order "less than" on the space of all ordered fuzzy numbers. These relations are defined as an extension of analogous relations on the space of all fuzzy numbers. All properties of the introduced relations have been investigated on the basis of the revised Kosiński's theory of ordered fuzzy numbers. It is shown here that in the above way these relations have been defined unambiguously as fuzzy ones. In addition, it is proven that the obtained relationships depend on the orientation of the difference between the compared numbers. The results obtained will be useful for formulating optimization tasks using numbers.*

Key words: *ordered fuzzy number, fuzzy relation, preorder, strict order*

JEL codes: *C52, C61, C651.*

1. Introduction

The intuitive concept of ordered fuzzy numbers was introduced by Kosiński and his co-workers (Kosiński et al., 2003, 2006) as an extension of the concept of fuzzy numbers (FN). OFNs' usefulness follows from the fact that an OFN is a FN supplemented by its orientation. Mentioned orientation may be negative or positive. Negative orientation means a linear order on $\mathbb{R}$ from bigger numbers to smaller ones. Negative oriented number is interpreted as such number, which may decrease. Positive orientation means a linear order on $\mathbb{R}$ from smaller numbers to bigger ones. Positive oriented number is interpreted as such number, which may increase. For formal reasons, the Kosiński' theory is revised (Piasecki, 2018) in this way that new definition of OFN fully corresponds to the intuitive defining it by Kosiński.

Unfortunately, the OFN theory has one significant drawback. This drawback is the lack of formal mathematical models dedicated to OFN issues. The purpose of this paper is to fill one of these theoretical gaps. For this reason, here our main aim is to introduce basic theory of ordering OFN.

2. Fuzzy numbers ordering – basic facts

For any space $\mathbb{A}$, the family of all its fuzzy subsets we will denote by the symbol $\mathcal{F}(\mathbb{A})$. An imprecise number is a family of values in which each considered value belongs to it in a varying degree. A commonly accepted model of imprecise number is the fuzzy number (FN), defined as a fuzzy subset of the real line $\mathbb{R}$. The most general definition of FN was given by Dubois and Prade (1979). Goetschel and Voxman (1986) have proved that the notion of FN may be equivalently defined in the following way.

Definition 1: Let for any nondecreasing sequence $\{a, b, c, d\} \subset \mathbb{R}$ the left reference function $L_{\mathcal{L}} \in [0; 1]^{[a,b]}$ and the right reference function $R_{\mathcal{L}} \in [0; 1]^{[c,d]}$ are upper semi-continuous monotonic functions satisfying the condition

$$L_{\mathcal{L}}(b) = R_{\mathcal{L}}(c) = 1. \tag{1}$$

Then the fuzzy subset $\mathcal{L}(a, b, c, d, L_{\mathcal{L}}, R_{\mathcal{L}}) \in \mathcal{F}(\mathbb{R})$ defined by its membership function $\mu_{\mathcal{L}} \in [0,1]^{\mathbb{R}}$:

$$\mu_{\mathcal{L}}(x) = \mu_{\mathcal{L}}(x|a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}}) = \begin{cases} 0, & x \notin [a,d] = [d,a], \\ L_{\mathcal{L}}(x), & x \in [a,b] = [b,a], \\ 1, & x \in [b,c] = [c,b], \\ R_{\mathcal{L}}(x), & x \in [c,d] = [d,c] \end{cases} \tag{2}$$ [1]

is called fuzzy number (FN).□

The space of all FNs we denote by the symbol $\mathbb{F}$. Any FN $\mathcal{L}(a,a,a,a,L_{\mathcal{L}},R_{\mathcal{L}})$ represents the crisp number $a \in \mathbb{R}$. Therefore, we can to write $\mathbb{R} \subset \mathbb{F}$.

Let us consider the pair $(\mathcal{K},\mathcal{L}) \in \mathbb{F} \times \mathbb{F}$ of FN represented respectively by their membership functions $\mu_{\mathcal{K}}, \mu_{\mathcal{L}} \in [0,1]^{\mathbb{R}}$. On the set $\mathbb{F}$ of all FNs we define the relation $\mathcal{K} \succcurlyeq \mathcal{L}$, which reads:

$$\text{“FN } \mathcal{K} \text{ is greater or equal to FN } \mathcal{L}.\text{”} \tag{3}$$

This relation is a fuzzy preorder $[Q] \in \mathcal{F}(\mathbb{F} \times \mathbb{F})$ determined by such membership function $\nu_{\boxed{Q}} \in [0,1]^{\mathbb{F}\times\mathbb{F}}$ which fulfils the condition (Orlovski, 1979):

$$\nu_{[Q]}(\mathcal{K},\mathcal{L}) = \sup\{\min\{\mu_{\mathcal{K}}(u), \mu_{\mathcal{L}}(v)\}: u \geq v\}. \tag{4}$$

From the point view of multivalued logic, the value $\nu_{[Q]}(\mathcal{K},\mathcal{L})$ may be interpreted as true-value of the sentence (3). If $\mathcal{L} = \mathcal{L}(0,0,0,0,L_{\mathcal{L}},R_{\mathcal{L}}) = 0$ then for any $\mathcal{K} = \mathcal{K}(a,b,c,d,L_{\mathcal{K}},R_{\mathcal{K}})$ we have:

$$\nu_{[Q]}(\mathcal{K},0) = \begin{cases} 0, & 0 > d, \\ R_{\mathcal{K}}(0), & d \geq 0 > c, \\ 1, & c \geq 0, \end{cases} \tag{5}$$

$$\nu_{[Q]}(0,\mathcal{K}) = \begin{cases} 1, & 0 \geq b, \\ L_{\mathcal{K}}(0), & b > 0 \geq a, \\ 0, & a > 0. \end{cases} \tag{6}$$

3. Ordered fuzzy numbers

The concept of ordered fuzzy numbers (OFN) was intuitively introduced by Kosiński and his co-writers in the series of papers (Kosiński et al, 2002), (Kosiński, 2006) as an extension of the concept of FN. The Kosiński's theory was revised in (Piasecki, 2018). In this paper, we will use the following revised definition of OFN.

Definition 2: Let for any monotonic sequence $\{a,b,c,d\} \subset \mathbb{R}$ the starting-function $L_{\mathcal{L}} \in [0;1]^{[a,b]}$ and the ending-function $R_{\mathcal{L}} \in [0;1]^{[c,d]}$ are upper semi-continuous monotonic functions satisfying the condition (1). Then the ordered fuzzy number (OFN) $\overleftrightarrow{\mathcal{L}}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$ is defined explicitly by its membership function $\mu_{\mathcal{L}}(\cdot\,|a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}}) \in [0,1]^{\mathbb{R}}$ given by (2). □

The space of all OFN we denote by the symbol $\mathbb{K}$. The condition $a < d$ fulfilment determines the positive orientation of OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$. In this case, the starting-function $L_{\mathcal{L}}$ is non-decreasing and the ending-function $R_{\mathcal{L}}$ is non-increasing. Any positively oriented OFN is interpreted as such imprecise number, which may increase. The space of all positively oriented OFN we denote by the symbol $\mathbb{K}^+$. The condition $a > d$ fulfilment determines the negative orientation of OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$. In this case, the starting-function $L_{\mathcal{L}}$ is non-increasing and the ending-function $R_{\mathcal{L}}$ is non-decreasing. Negatively oriented OFN is interpreted as such imprecise number, which may decrease. The space of all negatively oriented OFN we denote by the symbol $\mathbb{K}^-$. For the case $a = d$, OFN $\overleftrightarrow{\mathcal{L}}(a,a,a,a,L_{\mathcal{L}},R_{\mathcal{L}})$ represents crisp number $a \in \mathbb{R}$, which is not oriented. In summary, we can write:

$$\mathbb{K} = \mathbb{K}^+ \cup \mathbb{R} \cup \mathbb{K}^-. \tag{7}$$

For the case $a \geq d$ any FN $\mathcal{L}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$ is equal to OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$. On the other hand, for this case any OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$ is equal FN $\mathcal{L}(a,b,c,d,L_{\mathcal{L}},R_{\mathcal{L}})$. These facts imply that we have:

$$\mathbb{F} = \mathbb{K}^+ \cup \mathbb{R}, \tag{8}$$

$$\mathbb{K} = \mathbb{F} \cup \mathbb{K}^-. \tag{9}$$

The necessary arithmetic operators will generally be defined for OFN using the following concepts.

Definition 3: Cut-function $L^\star \in [u,v]^{[0;1]}$ of any upper semi-continuous nondecreasing function $L \in [0;1]^{[u;v]}$ is given by the identity:

[1] Let us note that this identity describes additionally extended notation of numerical intervals, which is used in this work.

$$L^{\star}(\alpha) = \min\{x \in [u, v]: L(x) \geq \alpha\}. \square \tag{10}$$

Definition 4: Cut-function $R^{\star} \in [u, v]^{[0;1]}$ of any upper semi-continuous nonincreasing function $R \in [0; 1]^{[u;v]}$ is given by the identity:

$$R^{\star}(\alpha) = \min\{x \in [u, v]: R(x) \geq \alpha\}. \square \tag{11}$$

Definition 5: Pseudo inverse function $l^{\lhd} \in [0; 1]^{[l(0),l(1)]}$ of any bounded continuous and nondecreasing function $l \in [l(0), l(1)]^{[0;1]}$ is given by the identity:

$$l^{\lhd}(x) = \max\{\alpha \in [0; 1]: l(\alpha) = x\}. \square \tag{12}$$

Definition 6: Pseudo inverse function $r^{\lhd} \in [0; 1]^{[r(1),r(0)]}$ of any bounded continuous and nonincreasing function $r \in [r(0), r(1)]^{[0;1]}$ is given by the identity:

$$r^{\lhd}(x) = \min\{\alpha \in [0; 1]: r(\alpha) = x\}. \square \tag{13}$$

Let us consider the OFNs $\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}, \overleftrightarrow{\mathcal{M}} \in \mathbb{K}$ described as follows:

$$\overleftrightarrow{\mathcal{K}} = \overleftrightarrow{\mathcal{K}}(a_{\mathcal{K}}, b_{\mathcal{K}}, c_{\mathcal{K}}, d_{\mathcal{K}}, L_{\mathcal{K}}, R_{\mathcal{K}}), \quad \overleftrightarrow{\mathcal{L}} = \overleftrightarrow{\mathcal{L}}(a_{\mathcal{L}}, b_{\mathcal{L}}, c_{\mathcal{L}}, d_{\mathcal{L}}, L_{\mathcal{L}}, R_{\mathcal{L}}), \quad \overleftrightarrow{\mathcal{M}} = \overleftrightarrow{\mathcal{M}}(a_{\mathcal{M}}, b_{\mathcal{M}}, c_{\mathcal{M}}, d_{\mathcal{M}}, L_{\mathcal{M}}, R_{\mathcal{M}}). \tag{14}$$

The addition operation " $+$ " on $\mathbb{R}$ is extended to addition operation $\boxplus$ on $\mathbb{K}$ by the identity (Piasecki, 2018):

$$\overleftrightarrow{\mathcal{M}} = \overleftrightarrow{\mathcal{K}} \boxplus \overleftrightarrow{\mathcal{L}}, \tag{15}$$

where we have:

$$\check{a}_{\mathcal{M}} = a_{\mathcal{K}} + a_{\mathcal{L}}, \tag{16}$$

$$b_{\mathcal{M}} = b_{\mathcal{K}} + b_{\mathcal{L}}, \tag{17}$$

$$c_{\mathcal{M}} = c_{\mathcal{K}} + c_{\mathcal{L}}, \tag{18}$$

$$\check{d}_{\mathcal{M}} = d_{\mathcal{K}} + d_{\mathcal{L}}, \tag{19}$$

$$a_{\mathcal{M}} = \begin{cases} \min\{\check{a}_{\mathcal{M}}, b_{\mathcal{M}}\}, & (b_{\mathcal{M}} < c_{\mathcal{M}}) \vee (b_{\mathcal{M}} = c_{\mathcal{M}} \wedge \check{a}_{\mathcal{M}} \leq \check{d}_{\mathcal{M}}), \\ \max\{\check{a}_{\mathcal{M}}, b_{\mathcal{M}}\}, & (b_{\mathcal{M}} > c_{\mathcal{M}}) \vee (b_{\mathcal{M}} = c_{\mathcal{M}} \wedge \check{a}_{\mathcal{M}} > \check{d}_{\mathcal{M}}), \end{cases} \tag{20}$$

$$d_{\mathcal{M}} = \begin{cases} \max\{\check{d}_{\mathcal{M}}, c_{\mathcal{M}}\}, & (b_{\mathcal{M}} < c_{\mathcal{M}}) \vee (b_{\mathcal{M}} = c_{\mathcal{M}} \wedge \check{a}_{\mathcal{M}} \leq \check{d}_{\mathcal{M}}), \\ \min\{\check{d}_{\mathcal{M}}, c_{\mathcal{M}}\}, & (b_{\mathcal{M}} > c_{\mathcal{M}}) \vee (b_{\mathcal{M}} = c_{\mathcal{M}} \wedge \check{a}_{\mathcal{M}} > \check{d}_{\mathcal{M}}). \end{cases} \tag{21}$$

$$\forall_{\alpha \in [0;1]}: \quad l_{\mathcal{M}}(\alpha) = \begin{cases} L_{\mathcal{K}}^{\star}(\alpha) + L_{\mathcal{L}}^{\star}(\alpha), & a_{\mathcal{M}} \neq b_{\mathcal{M}}, \\ b_{\mathcal{M}}, & a_{\mathcal{M}} = b_{\mathcal{M}}, \end{cases} \tag{22}$$

$$\forall_{\alpha \in [0;1]}: \quad r_{\mathcal{M}}(\alpha) = \begin{cases} R_{\mathcal{K}}^{\star}(\alpha) + R_{\mathcal{L}}^{\star}(\alpha), & c_{\mathcal{M}} \neq b_{\mathcal{M}}, \\ c_{\mathcal{M}}, & c_{\mathcal{M}} = d_{\mathcal{M}}. \end{cases} \tag{23}$$

$$\forall_{x \in [a_{\mathcal{M}}, b_{\mathcal{M}}]}: \quad L_{\mathcal{M}}(x) = l_{\mathcal{M}}^{\lhd}(x), \tag{24}$$

$$\forall_{x \in [c_{\mathcal{M}}, d_{\mathcal{M}}]}: \quad R_{M}(x) = r_{\mathcal{M}}^{\lhd}(x). \tag{25}$$

The unary minus operator "$-$" on $\mathbb{R}$ is extended to minus operator $\ominus$ on $\mathbb{K}$ by the identity:

$$\ominus \overleftrightarrow{\mathcal{K}} = \overleftrightarrow{\mathcal{K}^{(-)}}(-a_{\mathcal{K}}, -b_{\mathcal{K}}, -c_{\mathcal{K}}, -d_{\mathcal{K}}, L_{\mathcal{K}}^{(-)}, R_{\mathcal{K}}^{(-)}), \tag{26}$$

Where:

$$\forall_{x \in [-a_{\mathcal{K}}, -b_{\mathcal{K}}]}: \quad L_{\mathcal{K}}^{(-)}(x) = L_{\mathcal{K}}(-x), \tag{27}$$

$$\forall_{x \in [-c_{\mathcal{K}}, -d_{\mathcal{K}}]}: \quad R_{\mathcal{K}}^{(-)}(x) = R_{\mathcal{K}}(-x). \tag{28}$$

The subtraction operation " $-$ " on $\mathbb{R}$ is extended to subtraction operation $\boxminus$ on $\mathbb{K}$ by the identity:

$$\overleftrightarrow{\mathcal{M}} = \overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{L}} = \overleftrightarrow{\mathcal{K}} \boxplus (\ominus \overleftrightarrow{\mathcal{L}}). \tag{29}$$

Kosiński (2006) has shown that for any OFN $\overleftrightarrow{\mathcal{K}} \in \mathbb{K}$ we have:

$$\overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{K}} = 0. \tag{30}$$

4. Ordering of order fuzzy numbers

Let us consider the pair $(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) \in \mathbb{K} \times \mathbb{K}$ .On the set $\mathbb{K}$ of all OFNs we define the relation $\overleftrightarrow{\mathcal{K}} \succcurlyeq \overleftrightarrow{\mathcal{L}}$, which reads:

$$\text{“}\mathit{OFN}\ \overleftrightarrow{\mathcal{K}}\ \mathit{is\ greater\ or\ equal\ to\ OFN}\ \overleftrightarrow{\mathcal{L}}.\text{”} \tag{31}$$

This relation is a fuzzy preorder $Q \in \mathcal{F}(\mathbb{K} \times \mathbb{K})$ determined by such membership function $\nu_Q \in [0,1]^{\mathbb{K}\times\mathbb{K}}$ that from the point view of multivalued logic, the value $\nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}})$ may be interpreted as true-value of the sentence (31). Moreover, we assume that the membership function $\nu_Q \in [0,1]^{\mathbb{K}\times\mathbb{K}}$ fulfils the conditions:

- the law of parties' subtraction of inequality:

$$\forall_{(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}})\in\mathbb{K}\times\mathbb{K}}:\ \nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \nu_Q(\overleftrightarrow{\mathcal{K}}\boxminus\overleftrightarrow{\mathcal{L}},\overleftrightarrow{\mathcal{L}}\boxminus\overleftrightarrow{\mathcal{L}}) = \nu_Q(\overleftrightarrow{\mathcal{K}}\boxminus\overleftrightarrow{\mathcal{L}},0), \tag{32}$$

- the law of sign exchange:

$$\forall_{\overleftrightarrow{\mathcal{K}}\in\mathbb{K}}:\qquad \nu_Q(\overleftrightarrow{\mathcal{K}},0) = \nu_Q(0,\ominus\overleftrightarrow{\mathcal{K}}), \tag{33}$$

- the extension laws:

$$\forall_{\overleftrightarrow{\mathcal{K}}\in\mathbb{F}}:\qquad \nu_Q(\overleftrightarrow{\mathcal{K}},0) = \nu_{[Q]}(\overleftrightarrow{\mathcal{K}},0)\ \&\ \nu_Q(0,\overleftrightarrow{\mathcal{K}}) = \nu_{[Q]}(0,\overleftrightarrow{\mathcal{K}}). \tag{34}$$

Theorem 1: For any pair $(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) \in \mathbb{K} \times \mathbb{K}$ fulfilling the condition:

$$\overleftrightarrow{\mathcal{K}}\boxminus\overleftrightarrow{\mathcal{L}} = \overleftrightarrow{\mathcal{M}} = \overleftrightarrow{\mathcal{M}}(a_{\mathcal{M}},b_{\mathcal{M}},c_{\mathcal{M}},d_{\mathcal{M}},L_{\mathcal{M}},R_{\mathcal{M}}) \tag{35}$$

we have:

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{F}$ then:

$$\nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \begin{cases} 0, & 0 > d_{\mathcal{M}}, \\ R_{\mathcal{M}}(0), & d_{\mathcal{M}} \geq 0 > c_{\mathcal{M}}, \\ 1, & c_{\mathcal{M}} \geq 0, \end{cases} \tag{36}$$

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{K}^-$ then:

$$\nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \begin{cases} 0, & 0 > a_{\mathcal{M}}, \\ L_{\mathcal{M}}(0), & a_{\mathcal{M}} \geq 0 > b_{\mathcal{M}}, \\ 1, & b_{\mathcal{M}} \geq 0. \end{cases} \tag{37}$$

Proof: If $\overleftrightarrow{\mathcal{M}} \in \mathbb{F}$ then the identity (36) follows immediately from (32), (34) and (5). If $\overleftrightarrow{\mathcal{M}} \in \mathbb{K}^-$ then $(\ominus\overleftrightarrow{\mathcal{M}}) \in \mathbb{K}^+ \subset \mathbb{F}$. Therefore, using (32), (33), (34), (26) and (6) we obtain:

$$\nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \nu_Q(\overleftrightarrow{\mathcal{K}}\boxminus\overleftrightarrow{\mathcal{L}},0) = \nu_Q(\overleftrightarrow{\mathcal{M}},0) = \nu_Q(0,\ominus\overleftrightarrow{\mathcal{M}}) = \nu_{[Q]}(0,\ominus\overleftrightarrow{\mathcal{M}})$$

$$= \nu_{[Q]}\left(0,\overleftrightarrow{\mathcal{M}^{(-)}}(-a_{\mathcal{M}},-b_{\mathcal{M}},-c_{\mathcal{M}},-d_{\mathcal{M}},L_{\mathcal{M}}^{(-)},R_{\mathcal{M}}^{(-)})\right)$$

$$= \begin{cases} 1, & 0 \geq -b_{\mathcal{M}}, \\ L_{\mathcal{M}}^{(-)}(0), & -b_{\mathcal{M}} > 0 \geq -a_{\mathcal{M}}, \\ 0, & -a_{\mathcal{M}} > 0, \end{cases}$$

$$= \begin{cases} 0, & 0 > a_{\mathcal{M}}, \\ L_{\mathcal{M}}(0), & a_{\mathcal{M}} \geq 0 > b_{\mathcal{M}}, \\ 1, & b_{\mathcal{M}} \geq 0. \end{cases} \qquad \square$$

In the next step, we define the relation $\overleftrightarrow{\mathcal{K}} \succ \overleftrightarrow{\mathcal{L}}$, which reads:

$$\text{“}\mathit{OFN}\ \overleftrightarrow{\mathcal{K}}\ \mathit{is\ greater\ than\ OFN}\ \overleftrightarrow{\mathcal{L}}\text{”}. \tag{38}$$

This relation is a fuzzy strict order $S \in \mathcal{F}(\mathbb{K} \times \mathbb{K})$ determined by its membership function $\nu_S \in [0,1]^{\mathbb{K}\times\mathbb{K}}$ given by the identity:

$$\nu_S(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \min\{\nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}), 1 - \nu_Q(\overleftrightarrow{\mathcal{L}},\overleftrightarrow{\mathcal{K}})\}. \tag{39}$$

The value $\nu_S(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}})$ may be interpreted as true-value of the sentence (38).

Theorem 2: For any pair $(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) \in \mathbb{K}\times\mathbb{K}$ fulfilling the condition (35) we have:

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{F}$ then:

$$\nu_S(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \begin{cases} 1, & 0 < a_{\mathcal{M}}, \\ 1 - L_{\mathcal{M}}(0), & a_{\mathcal{M}} \le 0 < b_{\mathcal{M}}, \\ 0, & b_{\mathcal{M}} \le 0, \end{cases} \tag{40}$$

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{K}^-$ then:

$$\nu_S(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \begin{cases} 1, & 0 < d_{\mathcal{M}}, \\ 1 - R_{\mathcal{M}}(0), & d_{\mathcal{M}} \le 0 < c_{\mathcal{M}}, \\ 0, & c_{\mathcal{M}} \le 0. \end{cases} \tag{41}$$

Proof: If $\overleftrightarrow{\mathcal{M}} \in \mathbb{F}$, then from (32) and (37) we obtain:

$$1-\nu_Q(\overleftrightarrow{\mathcal{L}},\overleftrightarrow{\mathcal{K}}) = 1-\nu_Q(\ominus\overleftrightarrow{\mathcal{M}},0) = 1-\nu_Q\left(\overleftrightarrow{\mathcal{M}^{(-)}}(-a_{\mathcal{M}},-b_{\mathcal{M}},-c_{\mathcal{M}},-d_{\mathcal{M}},L_{\mathcal{M}}^{(-)},R_{\mathcal{M}}^{(-)}),0\right)$$

$$= \begin{cases} 1, & 0 > -a_{\mathcal{M}} \\ 1 - L_{\mathcal{M}}(0), & -a_{\mathcal{M}} \ge 0 > -b_{\mathcal{M}} \\ 0, & -b_{\mathcal{M}} \ge 0 \end{cases}$$

$$= \begin{cases} 1, & 0 < a_{\mathcal{M}}, \\ 1 - L_{\mathcal{M}}(0), & a_{\mathcal{M}} \le 0 < b_{\mathcal{M}}. \\ 0, & b_{\mathcal{M}} \le 0. \end{cases}$$

This result together with (36) implies:

$$\nu_S(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \min\{\nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}), 1-\nu_Q(\overleftrightarrow{\mathcal{L}},\overleftrightarrow{\mathcal{K}})\} = \begin{cases} 1, & 0 < a_{\mathcal{M}}, \\ 1 - L_{\mathcal{M}}(0), & a_{\mathcal{M}} \le 0 < b_{\mathcal{M}}, \\ 0, & b_{\mathcal{M}} \le 0. \end{cases}$$

If $\overleftrightarrow{\mathcal{M}} \in \mathbb{K}^-$, then from (32) and (36) we obtain:

$$1-\nu_Q(\overleftrightarrow{\mathcal{L}},\overleftrightarrow{\mathcal{K}}) = 1-\nu_Q(\ominus\overleftrightarrow{\mathcal{M}},0) = 1-\nu_Q\left(\overleftrightarrow{\mathcal{M}^{(-)}}(-a_{\mathcal{M}},-b_{\mathcal{M}},-c_{\mathcal{M}},-d_{\mathcal{M}},L_{\mathcal{M}}^{(-)},R_{\mathcal{M}}^{(-)}),0\right) =$$

$$= \begin{cases} 1, & 0 > -d_{\mathcal{M}}, \\ 1 - R_{\mathcal{M}}(0), & -d_{\mathcal{M}} \ge 0 > -c_{\mathcal{M}}, \\ 0, & -c_{\mathcal{M}} > 0 \end{cases}$$

$$= \begin{cases} 1, & 0 < d_{\mathcal{M}}, \\ 1 - R_{\mathcal{M}}(0), & d_{\mathcal{M}} \le 0 < c_{\mathcal{M}}, \\ 0, & c_{\mathcal{M}} \le 0. \end{cases}$$

This result together with (37) implies:

$$\nu_S(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}) = \min\{\nu_Q(\overleftrightarrow{\mathcal{K}},\overleftrightarrow{\mathcal{L}}), 1-\nu_Q(\overleftrightarrow{\mathcal{L}},\overleftrightarrow{\mathcal{K}})\} = \begin{cases} 1, & 0 < d_{\mathcal{M}}, \\ 1 - R_{\mathcal{M}}(0), & d_{\mathcal{M}} \le 0 < c_{\mathcal{M}}, \\ 0, & c_{\mathcal{M}} \le 0. \end{cases} \quad \square$$

For any finite set $A = \{\overleftrightarrow{\mathcal{K}}_1, \overleftrightarrow{\mathcal{K}}_2, \dots, \overleftrightarrow{\mathcal{K}}_n\} \subset \mathbb{K}$ we can distinguish its maximal element given as fuzzy subset $Max\{A\} \in \mathcal{F}(A)$. The maximal element $Max\{A\}$ is determined by its membership function $\mu_{Max\{A\}} \in [0,1]^A$ given by the identity (Orlovsky, 1979):

$$\mu_{Max\{A\}}(\overleftrightarrow{\mathcal{K}}_i) = \min\{\nu_Q(\overleftrightarrow{\mathcal{K}}_i,\overleftrightarrow{\mathcal{K}}_j): \overleftrightarrow{\mathcal{K}}_j \in A\}. \tag{42}$$

5. Trapezoidal ordered fuzzy numbers case

The main aim of the next section is to illustrate introduced above theory by means of numerical example. For calculation simplicity, in this example we will apply the trapezoidal OFN (TrOFN) defined as follows.

Definition 7: For any monotonic sequence $\{a,b,c,d\} \subset \mathbb{R}$, the trapezoidal OFN $\overleftrightarrow{Tr}(a,b,c,d)$ is determined explicitly by its membership functions $\mu_{Tr}(\cdot\,|a,b,c,d) \in [0,1]^{\mathbb{R}}$ as follows:

$$\mu_{Tr}(x|a,b,c,d) = \begin{cases} 0, & x \notin [a,d] = [d,a], \\ \frac{x-a}{b-a}, & x \in [a,b[\,=\,]b,a], \\ 1, & x \in [b,c] = [c,b], \\ \frac{x-d}{c-d}, & x \in]c,d] = [d,c[. \end{cases} \quad \square \tag{43}$$

It is obvious that any TrOFN is OFN. The orientation of TrOFN is determined in the same way as for any OFN. For any TrOFNs $\overleftrightarrow{\mathcal{K}} = \overleftrightarrow{Tr}(a_{\mathcal{K}}, b_{\mathcal{K}}, c_{\mathcal{K}}, d_{\mathcal{K}})$ and $\overleftrightarrow{\mathcal{L}} = \overleftrightarrow{Tr}(a_{\mathcal{L}}, b_{\mathcal{L}}, c_{\mathcal{L}}, d_{\mathcal{L}})$ we have:

$$\overleftrightarrow{\mathcal{M}} = \overleftrightarrow{Tr}(a_{\mathcal{M}}, b_{\mathcal{M}}, c_{\mathcal{M}}, d_{\mathcal{M}}) = \overleftrightarrow{Tr}(a_{\mathcal{K}}, b_{\mathcal{K}}, c_{\mathcal{K}}, d_{\mathcal{K}}) \boxed{+} \overleftrightarrow{Tr}(a_{\mathcal{L}}, b_{\mathcal{L}}, c_{\mathcal{L}}, d_{\mathcal{L}}), \tag{44}$$

where the sequence $\{a_M, b_M, c_M, d_M\}$ is explicitly determined by the identities (16)-(21). Moreover, any TrOFN $\overleftrightarrow{\mathcal{K}} = \overleftrightarrow{Tr}(a_{\mathcal{K}}, b_{\mathcal{K}}, c_{\mathcal{K}}, d_{\mathcal{K}})$ satisfies the condition:

$$\ominus \overleftrightarrow{Tr}(a_{\mathcal{K}}, b_{\mathcal{K}}, c_{\mathcal{K}}, d_{\mathcal{K}}) = \overleftrightarrow{Tr}(-a_{\mathcal{K}}, -b_{\mathcal{K}}, -c_{\mathcal{K}}, -d_{\mathcal{K}}). \tag{45}$$

TrOFNs can be equivalently defined as follows.

Definition 8: For any monotonic sequence $\{a, b, c, d\} \subset \mathbb{R}$, the trapezoidal OFN $\overleftrightarrow{Tr}(a, b, c, d)$ is determined explicitly by its membership functions $\mu_{Tr}(\cdot\,|a, b, c, d) \in [0,1]^{\mathbb{R}}$ as follows:

$$\mu_{Tr}(x|a, b, c, d) = \begin{cases} 0, & x \notin [a, d] = [d, a], \\ \mathcal{J}\left(\frac{x-a}{b-a}\right), & x \in [a, b[\, = \,]b, a], \\ 1, & x \in [b, c] = [c, b], \\ \mathcal{J}\left(\frac{x-d}{c-d}\right), & x \in\,]c, d] = [d, c[. \end{cases} \tag{46}$$

where the function $\mathcal{J} \in [0,1]^{[0,1]}$ is given by the identity:

$$\mathcal{J}(x) = x. \square \tag{47}$$

Then we can simplify the Theorems 1 and 2 in following way.

Theorem 3: For any pair $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in \mathbb{K} \times \mathbb{K}$ fulfilling the condition:

$$\overleftrightarrow{\mathcal{K}} \boxed{-} \overleftrightarrow{\mathcal{L}} = \overleftrightarrow{\mathcal{M}} = \overleftrightarrow{Tr}(a_{\mathcal{M}}, b_{\mathcal{M}}, c_{\mathcal{M}}, d_{\mathcal{M}}) \tag{48}$$

we have:

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{F}$ then:

$$\nu_Q(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \begin{cases} 0 & 0 > d_{\mathcal{M}}, \\ \frac{-d_{\mathcal{M}}}{c_{\mathcal{M}} - d_{\mathcal{M}}} & d_{\mathcal{M}} \geq 0 > c_{\mathcal{M}}, \\ 1 & c_{\mathcal{M}} \geq 0. \end{cases} \tag{49}$$

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{K}^-$ then:

$$\nu_Q(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \begin{cases} 0 & 0 > a_{\mathcal{M}}, \\ \frac{-a_{\mathcal{M}}}{b_{\mathcal{M}} - a_{\mathcal{M}}} & a_{\mathcal{M}} \geq 0 > b_{\mathcal{M}}, \\ 1 & b_{\mathcal{M}} \geq 0. \end{cases} \quad \square \tag{50}$$

Theorem 4: For any pair $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in \mathbb{K} \times \mathbb{K}$ fulfilling the condition (48) we have:

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{F}$ then:

$$\nu_S(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \begin{cases} 1 & 0 < a_{\mathcal{M}}, \\ \frac{b_{\mathcal{M}}}{b_{\mathcal{M}} - a_{\mathcal{M}}} & a_{\mathcal{M}} \leq 0 < b_{\mathcal{M}}, \\ 0 & b_{\mathcal{M}} \leq 0. \end{cases} \tag{51}$$

- if $\overleftrightarrow{\mathcal{M}} \in \mathbb{K}^-$ then:

$$\nu_S(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \begin{cases} 1 & 0 < d_{\mathcal{M}}, \\ \frac{c_{\mathcal{M}}}{c_{\mathcal{M}} - d_{\mathcal{M}}} & d_{\mathcal{M}} \leq 0 < c_{\mathcal{M}}, \\ 0 & c_{\mathcal{M}} \geq 0. \end{cases} \quad \square \tag{52}$$

6. Case study

Let us take into account finite set $A = \{\overleftrightarrow{\mathcal{K}}_1, \overleftrightarrow{\mathcal{K}}_2, \overleftrightarrow{\mathcal{K}}_3, \overleftrightarrow{\mathcal{K}}_4\} \subset$, where:

$$\overleftrightarrow{\mathcal{K}}_1 = \overleftrightarrow{\mathcal{L}}_1 = \overleftrightarrow{Tr}(10,20,30,40), \quad \overleftrightarrow{\mathcal{K}}_2 = \overleftrightarrow{\mathcal{L}}_2 = \overleftrightarrow{Tr}(5,25,40,60),$$

$$\overleftrightarrow{\mathcal{K}}_3 = \overleftrightarrow{\mathcal{L}}_3 = \overleftrightarrow{Tr}(15,15,10,5), \quad \overleftrightarrow{\mathcal{K}}_4 = \overleftrightarrow{\mathcal{L}}_4 = \overleftrightarrow{Tr}(25,15,10,5).$$

In the first step, for any pair $(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{L}}_j)$ of TrOFNs, we calculate their difference $\overleftrightarrow{\mathcal{M}}_{i,j} = \overleftrightarrow{\mathcal{K}}_i \boxminus \overleftrightarrow{\mathcal{L}}_j$. The results of these calculations are presented in table 1.

Tab. 1 Differences $\overleftrightarrow{\mathcal{M}}_{i,j}$ between compared TrOFN

	$\overleftrightarrow{\mathcal{K}}_1$	$\overleftrightarrow{\mathcal{K}}_2$	$\overleftrightarrow{\mathcal{K}}_3$	$\overleftrightarrow{\mathcal{K}}_4$
$\overleftrightarrow{\mathcal{L}}_1$	$\overleftrightarrow{Tr}(0,0,0,0)$	$\overleftrightarrow{Tr}(-5,5,10,20)$	$\overleftrightarrow{Tr}(5,-5,-20,-35)$	$\overleftrightarrow{Tr}(15,-5,-20,-25)$
$\overleftrightarrow{\mathcal{L}}_2$	$\overleftrightarrow{Tr}(5,-5,-10,-20)$	$\overleftrightarrow{Tr}(0,0,0,0)$	$\overleftrightarrow{Tr}(10,-10,-30,-55)$	$\overleftrightarrow{Tr}(20,-10,-30,-55)$
$\overleftrightarrow{\mathcal{L}}_3$	$\overleftrightarrow{Tr}(-5,5,20,35)$	$\overleftrightarrow{Tr}(-10,10,30,55)$	$\overleftrightarrow{Tr}(0,0,0,0)$	$\overleftrightarrow{Tr}(-15,-25,-25,-25)$
$\overleftrightarrow{\mathcal{L}}_4$	$\overleftrightarrow{Tr}(-15,5,20,25)$	$\overleftrightarrow{Tr}(-20,10,30,55)$	$\overleftrightarrow{Tr}(15,25,25,25)$	$\overleftrightarrow{Tr}(0,0,0,0)$

Source: Own calculation.

In the next, using identities (49) and (50), for any pair $(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{L}}_j)$ of TrOFNs, we calculate the membership function value $\nu_Q(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{L}}_j)$ evaluating the sentence $\overleftrightarrow{\mathcal{K}}_i \succcurlyeq \overleftrightarrow{\mathcal{L}}_j$. The results of these calculations are presented in Table 2. Moreover, we distinguish the maximal element $Max\{A\}$ determined by its membership function $\mu_{Max\{A\}} \in [0,1]^A$ given by the identity (42). The values $\mu_{Max\{A\}}(\overleftrightarrow{\mathcal{K}}_i)$ are shown in the last row of table 2. Let us note, that membership function $\mu_{Max\{A\}} \in [0,1]^A$ may be applied as a fuzzy choice function (Herrera and Herrera-Viedma, 2000).

Tab. 2 The membership functions $\nu_Q(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{L}}_j)$ of the relation $\overleftrightarrow{\mathcal{K}}_i \succcurlyeq \overleftrightarrow{\mathcal{L}}_j$ and $\mu_{Max\{A\}}(\overleftrightarrow{\mathcal{K}}_i)$ of maximal element $Max\{A\}$

	$\overleftrightarrow{\mathcal{K}}_1$	$\overleftrightarrow{\mathcal{K}}_2$	$\overleftrightarrow{\mathcal{K}}_3$	$\overleftrightarrow{\mathcal{K}}_4$
$\overleftrightarrow{\mathcal{L}}_1$	1	1	0.5	0.75
$\overleftrightarrow{\mathcal{L}}_2$	0.5	1	0.5	0.67
$\overleftrightarrow{\mathcal{L}}_3$	1	1	1	0
$\overleftrightarrow{\mathcal{L}}_4$	1	1	1	1
$\mu_{Max\{A\}}(\overleftrightarrow{\mathcal{K}}_i)$	0.5	1	0.5	0

Source: Own calculation.

In the last step, using the identities (51) and (52), for any pair $(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{L}}_j)$ of TrOFNs, we calculate the membership function value $\nu_S(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{L}}_j)$ evaluating the sentence $\overleftrightarrow{\mathcal{K}}_i \succ \overleftrightarrow{\mathcal{L}}_j$. These results are presented in table 3.

Tab. 3 The membership function $\nu_S(\overleftrightarrow{\mathcal{K}}_i, \overleftrightarrow{\mathcal{L}}_j)$ of the relation $\overleftrightarrow{\mathcal{K}}_i \succ \overleftrightarrow{\mathcal{L}}_j$

	$\overleftrightarrow{\mathcal{K}}_1$	$\overleftrightarrow{\mathcal{K}}_2$	$\overleftrightarrow{\mathcal{K}}_3$	$\overleftrightarrow{\mathcal{K}}_4$
$\overleftrightarrow{\mathcal{L}}_1$	0	0.5	0	0
$\overleftrightarrow{\mathcal{L}}_2$	0	0	0	0
$\overleftrightarrow{\mathcal{L}}_3$	0.5	0.5	0	0
$\overleftrightarrow{\mathcal{L}}_4$	0.25	0.33	1	0

Source: Own calculation.

6. Conclusions

The OFNs' ordering obtained here will allow us to use the financial ratios proposed in (Łyczkowska-Hanćkowiak and Piasecki, 2018; Piasecki, 2017) in the investment decision making models described in (Piasecki, 2011, 2014). Moreover, this ordering may be applied for practical problems described in (Kacprzak et al., 2013; Prokopowicz et al., 2017; Roszkowska and Kacprzak 2016).

Unfortunately, the OFN linear space has one annoying drawback. Very difficult problem is the high computational complexity of the task of determining the difference between the compared OFNs. This results in the fact that undertaking research on the simplification of this task will allow intensive application of the proposed ordering relations.

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