ROY'S CRITERION DETERMINED BY ORIENTED FUZZY DISCOUNT FACTOR

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Abstract: *The subject of the analysis will be security with present value evaluated by ordered fuzzy number. This means that present value was estimated in an imprecise manner and supplemented by the forecast of its future changes. This present value is called oriented fuzzy present value. The discount factor of such security is ordered fuzzy number. All classic portfolio analysis methods are based on the concept of a return rate. On the other hand, the literature showed that in the case of securities with fuzzy present value, the discount factor is a better tool for portfolio analysis than the return rate. This raises the postulate of re-writing selected security management methods to equivalent methods based on discount factor. This will allow the application of these methods to the case of a security with present value evaluated by ordered fuzzy number. In this work an exemplary result of the implementation of the above postulate is presented. The main purpose of the presented article is to generalize the Roy's criterion to the case of management of investment recommendations formulated towards security characterized by the oriented discount factor. A five-grade scale rating has been used here.*

Key words: *first safety criteria, Roy's criterion, ordered fuzzy number, fuzzy oriented discount factor*
JEL codes: *C44, C02, G10*

1. Introduction

Present value (PV) is defined in (Piasecki, 2012) as a present equivalent of a payment available in a given time in the future. PV of future cash flows is widely accepted to be an approximate value, with fuzzy numbers being one of the main tools of its modelling.

If present value is evaluated by a fuzzy number, then the return rate is considered as a fuzzy probabilistic set (Piasecki, 2011). The expected return rate is obtained as a fuzzy subset in the real line. This result is a theoretical foundation for investment strategies presented in (Piasecki, 2014, 2018a). In (Piasecki, 2016) these results are generalized for the case when the present value is evaluated by intuitionistic fuzzy number (Atanassov, 1986).

Ordered fuzzy numbers (OFN) are defined in an intuitive way by Kosiński and his collaborators which in this way were going to introduce a fuzzy number supplemented by the orientation (Kosiński et al, 2002, 2006). The positive orientation means the expectation of the value growth, the negative one forecasts the decrease in the value. The competent elaboration on the current state-of-art on OFNs is the monograph (Prokopowicz et al, 2017). Kosiński has shown that there exist such OFNs which are not fuzzy numbers (Kosiński, 2006). For this reason, the original Kosiński's theory was revised in (Piasecki, 2018b).

The main aim of the paper is the analysis of the possibility to expand the above-mentioned investment strategies to a case when the present value is examined via ordered fuzzy numbers. To fulfil that task, the Safety-First Criterion – Roy's Criterion (Roy, 1952) will be extended for that case.

In the original Roy's Criterion, the main premise for formulating investment recommendation is the expected return rate of the analysed security. On the other hand, in (Piasecki & Siwek, 2018a, 2018b) it is shown that the expected fuzzy discount factor is a better tool for appraising considered securities than the fuzzy expected return rate. It was shown that the use of the expected fuzzy discount factor significantly facilitates the portfolio analysis. For this reason, the original Sharpe's ratio will be transformed to an equivalent form in which a basic premise to form investment recommendation is the expected fuzzy discount factor in case of analysed securities.

The paper is organised as follows. Chapter 2 presents the basic definitions and characteristics of OFNs. Chapters 3 and 4 briefly discuss oriented fuzzy present value and oriented fuzzy discount factor respectively. In Chapter 5 a five-step scale of investment recommendation was presented. Chapter 6 describes Roy's Criterion for the oriented discount factor. This will enable the use of a modified criterion for investment recommendations management. Chapter 7 includes an example which illustrates the use of predefined Roy's Criterion for management of investment recommendations with the present value given as trapezoidal oriented fuzzy number. The last Chapter is a brief summary.

2. Ordered fuzzy numbers – basic facts

By $\mathcal{F}(\mathbb{R})$ we denote the family of all fuzzy subsets of a real line $\mathbb{R}$. An imprecise number is a family of values in which each considered value belongs to it in a varying degree. A commonly accepted model of the imprecise number is the fuzzy number (FN), defined as a fuzzy subset of the real line $\mathbb{R}$. The most general definition of FN was given by Dubois and Prade (1978). Fuzzy number is defined in a following manner

Definition 1: For any FN $\mathcal{L}$ there exists such nondecreasing sequence $(a, b, c, d) \subset \mathbb{R}$ that $\mathcal{L} \in \mathcal{F}(\mathbb{R})$ is represented by its membership function $\mu_{\mathcal{L}}(\cdot\,|a, b, c, d, L_L, R_L) \in [0;1]^{\mathbb{R}}$ given by the identity

$$\mu_{\mathcal{L}}(x|a, b, c, d, L_L, R_L) = \begin{cases} 0, & x \notin [a, d], \\ L_L(x), & x \in [a, b], \\ 1, & x \in [b, c], \\ R_L(x), & x \in [c, d], \end{cases} \tag{1}$$

where the left reference function $L_L \in [0;1]^{[a,b]}$ and the right reference function $R_L \in [0;1]^{[c,d]}$ are the upper semi-continuous monotonic functions satisfying the conditions

$$L_L(b) = R_L(c) = 1, \tag{2}$$

$$\forall_{x\in]a,d[} : \ \mu_{\mathcal{L}}(x|a, b, c, d, L_L, R_L) > 0\ . \ \square \tag{3}$$

The family of all FN is denoted as $\mathbb{F}$. Moreover, Dubois and Prade (1979) have introduced such arithmetic operations on FN which are coherent with the Zadeh Extension Principle.

Ordered fuzzy numbers (OFN) were intuitively introduced by Kosiński and his co-writers in the series of papers (Kosiński et al, 2002, 2006) as an extension of the concept of FN. A significant drawback of Kosiński's theory is that there exist such OFNs which are not FN (Kosiński, 2006). The intuitive Kosiński's approach to the notion of OFN is very useful. The OFNs' usefulness results from the fact that an OFN is defined as FN supplemented by a negative orientation or a positive one. The negative orientation means the order from bigger to smaller numbers. The positive orientation means the order from smaller to bigger numbers. The FN orientation is interpreted as prediction of future FN changes. The Kosiński's theory was revised by Piasecki in (Piasecki, 2018b). OFNs are generally defined in following way:

Definition 2: Let for any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$ an ordered fuzzy number (OFN) $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ be defined as a pair of fuzzy numbers determined by their membership function $\mu_{\mathcal{L}}(\cdot\,|a, b, c, d, S_L, E_L) \in [0;1]^{\mathbb{R}}$ given by the identity

$$\mu_{\mathcal{L}}(x|a, b, c, d, S_L, E_L) = \begin{cases} 0, & x \notin [a, d] = [d, a], \\ S_L(x), & x \in [a, b] = [b, a], \\ 1, & x \in [b, c] = [c, b], \\ E_L(x), & x \in [c, d] = [d, c] \end{cases} \tag{4}$$

and orientation $[\![a \rightarrowtail d]\!] = (a, d)$, where the starting-function $S_L \in [0;1]^{[a,b]}$ and the ending-function $E_L \in [0;1]^{[c,d]}$ are upper semi-continuous monotonic functions satisfying the conditions

$$S_L(b) = E_L(c) = 1. \tag{5}$$

$$\forall_{x \in]a,d[} \quad \mu_{\mathcal{L}}(x|a, b, c, d, S_L, E_L) > 0\ .\square \tag{6}$$

Let us note that the identity (4) describes the additionally extended notation of numerical intervals, which is used in this work.

The space of all OFN is denoted by the symbol $\mathbb{K}$. The condition $a < d$ fulfilment determines the positive orientation $[\![a \rightarrowtail d]\!]$ of OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$. In this case, the starting-function S_L is non-decreasing and the ending-function E_L is non-increasing. Any positively oriented OFN is interpreted as such an imprecise number, which may increase. The condition $a > d$ fulfilment determines the negative orientation of OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$. In this case, the starting-function S_L is non-increasing and the ending-function E_L is non-decreasing. Negatively oriented OFN is interpreted as such an imprecise number, which may decrease. For the case $a = d$, OFN $\overleftrightarrow{\mathcal{L}}(a, a, a, a, S_L, E_L)$ represents a crisp number $a \in \mathbb{R}$, which is not oriented.

Herein, we will limit our deliberations to a special case of ordered fuzzy numbers – trapezoidal ordered fuzzy numbers defined in (Piasecki, 2018b) in a following manner.

Definition 3. For any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$ the trapezoidal ordered fuzzy number (TrOFN) $\overleftrightarrow{Tr}(a, b, c, d)$ is defined as the pair of FNs determined by their membership function $\mu_{\overleftrightarrow{Tr}}(\cdot\,|a, b, c, d) \in [0;1]^{\mathbb{R}}$ given by the identity

$$\mu_{\overleftrightarrow{Tr}}(x|a, b, c, d) = \begin{cases} 0, & x \notin [a, d] = [d, a], \\ \frac{x-a}{b-a}, & x \in [a, b[\, = \,]b, a], \\ 1, & x \in [b, c] = [c, b], \\ \frac{x-d}{c-d}, & x \in \,]c, d] = [d, c[\end{cases} \tag{7}$$

and orientation $[\![a \rightarrowtail d]\!]$. $\square$

The space of all TrOFNs will be denotes by the symbol $\mathbb{K}_{Tr}$.

Kosiński has introduced the arithmetic operators of dot product $\odot$ for TrOFNs in a following way. For any real number $\beta \in \mathbb{R}$ and any TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$ their dot product can be calculated as follows:

$$\beta \odot \overleftrightarrow{Tr}(a,b,c,d) = \overleftrightarrow{Tr}(\beta \cdot a, \beta \cdot b, \beta \cdot c, \beta \cdot d). \tag{8}$$

Also, the arithmetic proposed by Kosiński has a significant disadvantage. The space of ordered fuzzy numbers is not closed under Kosiński's addition. Therefore, the Kosinski's theory is modified in this way that the space of ordered fuzzy numbers is closed under revised arithmetic operations. The sum $\boxplus$ for TrOFNs is determined as follows (Piasecki, 2018b). In case of any TrOFNs $\overleftrightarrow{Tr}(a,b,c,d)$ and $\overleftrightarrow{Tr}(p-a,q-b,r-c,s-d)$ their sum is determined as follows:

$$\overleftrightarrow{Tr}(a,b,c,d) \boxplus \overleftrightarrow{Tr}(p-a,q-b,r-c,s-d) =$$
$$= \begin{cases} \overleftrightarrow{Tr}(\min\{p,q\},q,r,\max\{r,s\}), & (q<r) \vee (q=r \wedge p \le s), \\ \overleftrightarrow{Tr}(\max\{p,q\},q,r,\min\{r,s\}), & (q>r) \vee (q=r \wedge p>s). \end{cases} \tag{9}$$

In the set of trapezoidal ordered fuzzy numbers, the relation of a fuzzy preorder (Piasecki, 2018a) was determined. Let us consider the pair $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in \mathbb{K}_{Tr} \times \mathbb{K}_{Tr}$. On the set $\mathbb{K}_{tr}$ of all OFNs we define the relation $\overleftrightarrow{\mathcal{K}} \succcurlyeq \overleftrightarrow{\mathcal{L}}$ as follows:

$$\overleftrightarrow{\mathcal{K}} \succcurlyeq \overleftrightarrow{\mathcal{L}} \Leftrightarrow \text{“}TrOFN\ \overleftrightarrow{\mathcal{K}}\ is\ greater\ or\ equal\ to\ TrOFN\ \overleftrightarrow{\mathcal{L}}.\text{”} \tag{10}$$

This relation is a fuzzy preorder $Q \in \mathcal{F}(\mathbb{K}_{Tr} \times \mathbb{K}_{Tr})$ determined by means of such membership function $\nu_Q \in [0,1]^{\mathbb{K}_{Tr} \times \mathbb{K}_{Tr}}$ that from the point of view of multivalued logic, the value $\nu_Q(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}})$ may be interpreted as true-value of the sentence (10). The variability of membership function ν_Q is described in detail as follows:

Theorem 1 For any pair $(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) \in \mathbb{K} \times \mathbb{K}$ satisfying the condition

$$\overleftrightarrow{\mathcal{K}} \boxplus \left((-1) \odot \overleftrightarrow{\mathcal{L}}\right) = \overleftrightarrow{\mathcal{M}} = \overleftrightarrow{Tr}(a_{\mathcal{M}}, b_{\mathcal{M}}, c_{\mathcal{M}}, d_{\mathcal{M}}) \tag{11}$$

we have:

- if $a_{\mathcal{M}} \le d_{\mathcal{M}}$ then

$$\nu_Q(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \begin{cases} 0, & 0 > d_{\mathcal{M}}, \\ \frac{-d_{\mathcal{M}}}{c_{\mathcal{M}} - d_{\mathcal{M}}}, & d_{\mathcal{M}} \ge 0 > c_{\mathcal{M}}, \\ 1, & c_{\mathcal{M}} \ge 0, \end{cases} \tag{12}$$

- if $a_{\mathcal{M}} > d_{\mathcal{M}}$ then

$$\nu_Q(\overleftrightarrow{\mathcal{K}}, \overleftrightarrow{\mathcal{L}}) = \begin{cases} 0, & 0 > a_{\mathcal{M}}, \\ \frac{-a_{\mathcal{M}}}{b_{\mathcal{M}} - a_{\mathcal{M}}}, & a_{\mathcal{M}} \geq 0 > b_{\mathcal{M}}. \\ 1, & b_{\mathcal{M}} \geq 0. \end{cases} \qquad \square \tag{13}$$

3. Oriented fuzzy present value

The present value (PV) is called the current equivalent value of payments at a fixed point in time (Piasecki, 2012). The present value of the future cash flow may be imprecise. For this reason, the PV is described using fuzzy numbers. Then PV is characterized by a monotonic sequence $\{V_s, V_f, \check{C}, V_l, V_e\}$, where:

- $\check{C}$ – market price,
- $[V_s, V_e] \subset \mathbb{R}^+$ is an interval of all possible PV values,
- $[V_f, V_l] \subset [V_s, V_e]$ is an interval of all prices which do not perceptibly differ from the market price $\check{C}$.

PV was estimated in an imprecise manner and it was supplemented by a forecast of its closest changes. Such a present value is called the oriented present value (OPV). OPV is estimated by OFN:

$$\overleftrightarrow{PV} = \overleftrightarrow{S}(V_s, V_f, V_l, V_e, L_{PV}, R_{PV}), \tag{14}$$

where the left reference function $L_{PV}: [V_s; V_f] \to [0;1]$ and the right reference function R_{PV}: $[V_l; V_e] \to [0;1]$ are given. If we predict a rise in the market price, then OPV is described by a positively oriented OFN. If we predict a fall in market price, then OPV is described by a negatively oriented OFN. In this paper OPV is approximated by TrOFN

$$\overleftrightarrow{PV} = \overleftrightarrow{Tr}(V_s, V_f, V_l, V_e). \tag{15}$$

If $V_s < V_e$, then the positively oriented PV means the forecast of the value increase. If $V_s < V_e$, then the negative orientation is the forecast of the decrease in value.

Example 1 (Łyczkowska-Hanćkowiak, 2019): We evaluate the portfolio π composed of blocks of shares in Assecopol (ACP), ENERGA (ENG), JSW (JSW), KGHM (KGH), LOTOS (LTS), ORANGEPL (OPL) and PKOBP (PKO). Based on closing of the session on the Warsaw Stock Exchange on January 15, 2018, for each considered share we determine its OPV as TrOFN describing its Japanese candle. Obtained in this way shares' OPVs are presented in Table 1. For each portfolio component, we determine its market price $\check{C}_s$ as initial price on 16.01.2018.

Tab. 1. Evaluation of portfolio π components.

Stock Company	Present Value $\overleftrightarrow{PV}_s$	Market Price $\check{C}_s$	Expected Return Rate $\bar{r}_s$	Variance σ_s^2
ACP	$\overleftrightarrow{\mathcal{Tr}}(45.90;\ 45.90;\ 45.50;\ 45.48)$	45.70	0.0300	0.000090
CPS	$\overleftrightarrow{\mathcal{Tr}}(22.92;\ 22.82;\ 22.82;\ 22.76)$	22.82	0.0355	0.000190
ENG	$\overleftrightarrow{\mathcal{Tr}}(10.22;\ 10.19;\ 10.17; 10.14)$	10.18	0.0150	0.000020
JSW	$\overleftrightarrow{\mathcal{Tr}}(92.24;\ 92.54; 92.54; 92.80)$	92.54	0.0400	0.000290
KGH	$\overleftrightarrow{\mathcal{Tr}}(102.65;\ 103.05; 103.60;\ 103.90)$	103.33	0.0390	0.000210
LTS	$\overleftrightarrow{\mathcal{Tr}}(56.70;\ 56.56;\ 56.40;\ 56.28)$	56.48	0.0450	0.000390
OPL	$\overleftrightarrow{\mathcal{Tr}}(5.75;\ 5.76;\ 5.90;\ 5.90)$	5.83	0.0360	0.000280
PGE	$\overleftrightarrow{\mathcal{Tr}}(10.39;\ 10.39; 10.35;\ 10.33)$	10.37	0.0235	0.000160
PKO	$\overleftrightarrow{\mathcal{Tr}}(42.61;\ 42.61;\ 43.22;\ 43.22)$	42.91	0.0420	0.000370

Source Łyczkowska-Hanćkowiak, 2019 and own elaboration.

We notice that the stock companies KGH, JWS, OPL and PKO are evaluated by positively oriented PV, which predicts a rise in market price. Similarly the stock companies ACP, CPS, ENG, LTS and PGE are evaluated by negatively oriented PV, which predicts a fall in market price.

4. Oriented fuzzy discount factor

Let us assume that the time horizon $t > 0$ of an investment is fixed. Then, the security considered here is determined by two values: anticipated $FV = V_t$ and assessed $PV = V_0$. The basic characteristic of benefits from owning this security is the simple return rate defined as:

$$r_t = \frac{V_t - V_0}{V_0} = \frac{V_t}{V_0} - 1. \tag{16}$$

In practice of financial markets analysis, the uncertainty risk is usually described by probability distribution of return rate calculated for $V_0 = \check{C}$. Then the expected discount factor (EDF) $\bar{v} \in \mathbb{R}$ is given by the identity:

$$\bar{v} = \frac{1}{1+\bar{r}}. \tag{17}$$

Example 2 (Łyczkowska-Hanćkowiak, 2019) All considerations in the paper are run for the quarterly period of investment time $t = 1\,quarter$. Expected return rates of portfolio components π are presented in Table 1.

Oriented expected discount factor (OEDF) described by TrOFN was given in (Łyczkowska-Hanćkowiak & Piasecki, 2018)

$$\overleftrightarrow{V} = \overleftrightarrow{Tr}\left(\frac{V_s}{\check{C}} \cdot \bar{v}, \frac{V_f}{\check{C}} \cdot \bar{v},\ \frac{V_l}{\check{C}} \cdot \bar{v}, \frac{V_e}{\check{C}} \cdot \bar{v}\right). \tag{18}$$

Example 3: Using (18), we calculated quarterly EDF and OEDF for each component of the portfolio π. Obtained in this way evaluations are presented in Table 2 (Łyczkowska-Hanćkowiak & Piasecki, 2018).

The discount factor of a security described in this way is an oriented fuzzy number with the identical orientation as the oriented present value use for estimation. It is worth stressing that the maximum criterion of the expected return rate can be replaced by the minimum criterion of the expected discount factor. In case of fuzzy values of both parameters, those criteria are equivalentSymbols denoting vectors and matrices should be indicated in bold type. Scalar variable names should normally be expressed using italics. All non-standard abbreviations or symbols.

5. Investment recommendations

The investment recommendation is the counsel given by the advisors to the investor. In this paper we will consider the family of advice which is applied in (Piasecki, 2014). Therefore, recommendations are expressed by means of standardized advice (Piasecki, 2014):

- Buy – suggesting that evaluated security is significantly undervalued,
- Accumulate – suggesting that evaluated security is undervalued,
- Hold – suggesting that evaluated security is fairly valued,
- Reduce – suggesting that evaluated security is overvalued,
- Sell – suggesting that evaluated security is significantly overvalued.

The above-mentioned advice constitutes the set $\mathbb{A} = \{A^{++}, A^{+}, A^{0}, A^{-}, A^{--}\}$ - called a rating scale where

- A^{++} denotes the advice Buy,
- A^{+} denotes the advice Accumulate,
- A^{0} denotes the advice Hold,
- A^{-} denotes the advice Reduce,
- A^{--} denotes the advice Sell.

Due to such approach we will be able to compare the obtained results with the results of similar research conducted in (Piasecki, 2014, 2016).

Let us take into account a fixed security $\check{S}$, represented by the pair $(\bar{r}_s, \varpi_s)$ where $\bar{r}_s$ is an expected return on $\check{S}$ and ϖ_s is a parameter characterizing the security $\check{S}$. Adviser's counsel depends on the expected return. The criterion for a competent choice of advice can be presented as a comparison of the values profit $g(\bar{r}_s|\varpi_s)$ and the profitability threshold (PT) $\check{G}$, where $g(\cdot|\varpi_s)\colon \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function of the expected return rate. By the symbol $\mathbb{S}$ we denote the set of all considered securities. The advice choice function $\Lambda\colon \mathbb{S} \times \mathbb{R} \rightarrow 2^{\mathbb{A}}$ was given in (Piasecki, 2014). This way was assigned the advice subset $\Lambda(\check{S}, \check{G}) \subset \mathbb{A}$ which is interpreted as the investment recommendation given for the security. We can assume that a given security $\check{S}$ is represented by the ordered pair $(\bar{v}_s, \varpi_s)$ where $\bar{v}_s = \frac{1}{1+\bar{r}_s}$ is the expected discount factor (EDF) for $\check{S}$. Then [Łyczkowska-Hanćkowiak, 2019],

$$g(\bar{r}_s|\varpi_s) \geq \check{G} \Leftrightarrow \bar{v}_s \leq \frac{1}{1+g^{-1}(\check{G}|\varpi_s)} = H_s, \tag{19}$$

$$g(\bar{r}_s|\varpi_s) \leq \check{G} \Leftrightarrow \bar{v}_s \geq H_s. \tag{20}$$

The value H_s is interpreted as a specific profitability threshold (SPT) determined for the security $\check{S}$ Then advice choice function $\Lambda\colon \mathbb{S} \times \mathbb{R} \rightarrow 2^{\mathbb{A}}$ is equivalently described in a following way

- $A^{++} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \leq H_s \wedge \neg \bar{v}_s \geq H_s,$
- $A^{+} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \leq H_s,$
- $A^{0} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \leq H_s \wedge \bar{v}_s \geq H_s,$ (21)
- $A^{-} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \geq H_s,$
- $A^{--} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \neg \bar{v}_s \leq H_s \wedge \bar{v}_s \geq H_s.$

We consider the case when EDFs are imprecisely evaluated. Moreover, we can additively predict future changes in EDF value. Then a given security $\check{S}$ is represented by the ordered pair $(\overleftrightarrow{V}_s, \varpi_s)$ where

$$\overleftrightarrow{V}_s = \overleftrightarrow{Tr}\left(D_b^{(s)}, D_f^{(s)}, D_l^{(s)}, D_b^{(s)}\right) \tag{22}$$

is OEDF calculated with use (18) for $\check{S}$. For this case, using (19) we calculate specific profitability threshold H_s. If the PT $\check{G}$ is given, then each SPT H_s may be represented by TrOFN

$$\overleftrightarrow{[\![H_s]\!]} = \overleftrightarrow{Tr}(H_s, H_s, H_s, H_s). \tag{23}$$

Then the value $\tilde{\Lambda}(\check{S}, \check{G})$ of the recommendation choice function $\tilde{\Lambda}\colon [0,1]^{\mathbb{R}} \times \mathbb{R} \rightarrow [0,1]^{\mathbb{R}}$ is the membership function $\lambda(\cdot|\check{S}, \check{G})\colon \mathbb{A} \rightarrow [0,1]$ function determined in accordance with (21) in the following way:

- $\lambda(A^{++}|\check{S}, \check{G}) = \nu_Q\left(\overleftrightarrow{[\![H_s]\!]}, \overleftrightarrow{V}_s\right) \wedge \left(1 - \nu_Q\left(\overleftrightarrow{V}_s, \overleftrightarrow{[\![H_s]\!]}\right)\right),$
- $\lambda(A^{+}|\check{S}, \check{G}) = \nu_Q\left(\overleftrightarrow{[\![H_s]\!]}, \overleftrightarrow{V}_s\right),$

- $\lambda(A^0|\check{S},\check{G}) = \nu_Q\left(\overleftrightarrow{[\![H_s]\!]},\overleftrightarrow{V_s}\right) \wedge \nu_Q\left(\overleftrightarrow{V_s},\overleftrightarrow{[\![H_s]\!]}\right)$, (24)
- $\lambda(A^-|\check{S},\check{G}) = \nu_Q\left(\overleftrightarrow{V_s},\overleftrightarrow{[\![H_s]\!]}\right)$,
- $\lambda(A^{--}|\check{S},\check{G}) = \nu_Q\left(\overleftrightarrow{V_s},\overleftrightarrow{[\![H_s]\!]}\right) \wedge \left(1-\nu_Q\left(\overleftrightarrow{[\![H_s]\!]},\overleftrightarrow{V_s}\right)\right)$.

From the point of view of a multivalued logic, the value $\lambda(A|\check{S},\check{G})$ is interpreted as a logic value of the sentence

$$"Recommendation\ A \in \mathbb{A}\ is\ advised\ to\ take\ an\ investment\ decision". \tag{25}$$

From the point-view of decision-making, the value $\lambda(A|\check{S},\check{G})$ is interpreted as a degree of recommendation support $A \in \mathbb{A}$, i.e. a declared share of the advisor in the responsibility in case of final decision-making according to the advice $A \in \mathbb{A}$. In the described situation the investment recommendation $\tilde{\Lambda}(\check{S},\check{G})$ is the fuzzy subset in the rating scale $\mathbb{A}$..

6. The Roy's Criterion

Roy (1952) has consider a fixed security $\check{S}$, represented by the pair $(\bar{r}_s, \sigma_s)$, where $\bar{r}_s$ is an expected return on $\check{S}$ and σ_S^2 is the variance of return rate from considered financial instrument. After Markowitz (1952) we assume that considered security $\check{S}$ has simple return rate with Gaussian distribution $N(\bar{r}_s, \sigma_s)$.This distribution is described by its increasing and continuous cumulative distribution function $F(\cdot\,|\bar{r}_s, \sigma_s)\colon \mathbb{R} \to [0;1]$ given by the identity

$$F(x|\bar{r}_s,\sigma_s) = \Phi\left(\frac{x-\bar{r}_s}{\sigma_s}\right), \tag{26}$$

where the function $\Phi\colon \mathbb{R} \to [0;1]$ is the cumulative distribution function of the Gaussian distribution $N(0,1)$

The Safety Condition (Roy, 1952) is given as follows:

$$F(L|\bar{r}_s,\sigma_s) = \varepsilon, \tag{27}$$

where

- L – minimum acceptable return rate,
- ε – probability realization of return below the minimum acceptable rate.

The realization of return below the minimum acceptable rate is identified with a loss. The Roy criterion minimizes the probability of loss for a set minimum acceptable rate of return (Piasecki, 2014). Additionally in order to ensure financial security, the investor assumes the maximum level ε^* of loss probability. Then the Roy's criterion is described by the inequality

$$F(L|\bar{r}_s,\sigma_s) = \Phi\left(\frac{L-\bar{r}_s}{\sigma_s}\right) \le \varepsilon^* < \frac{1}{2}. \tag{28}$$

It implies that

$$\bar{r}_s \geq L - \sigma_s \cdot \Phi^{-1}(\varepsilon^*). \tag{29}$$

In line with (19), SPT is given as follows

$$H_s = \frac{1}{1+L-\sigma_s \cdot \Phi^{-1}(\varepsilon^*)}. \tag{30}$$

7. Case Study

In this section, we present recommendations based on Roy criterion for portfolio π components described in Example 1. Imprecise evaluations of PV and market price of those assets are presented on Table 1.

Investor takes into account the minimal acceptable return rate $L = 0.0075$. Additionally in order to ensure financial security, the investor assumes the maximum level of loss probability $\varepsilon^* = 0.05$. Then we have $\Phi^{-1}(0.05) = -1.64$.

Table 2 lists the values of EDF, OEDF and SPT determined for each components of portfolio π. These values are the only premises to formulate the investment recommendations.

Tab. 2 Expected discount factors of portfolio π components.

Stock Company	EDF $\bar{v}_s$	OEDF $\overleftrightarrow{V}_s$	SPT H_s
ACP	0.9709	$\overleftrightarrow{\mathcal{Tr}}(0.9751; 0.9751; 0.9666; 0.9662)$	0.9775
CPS	0.9657	$\overleftrightarrow{\mathcal{Tr}}(0.9699; 0.9657; 0.9657; 0.9632)$	0.9708
ENG	0.9852	$\overleftrightarrow{\mathcal{Tr}}(0.9891; 0.9862; 0.9842; 0.9813)$	0.9854
JSW	0.9615	$\overleftrightarrow{\mathcal{Tr}}(0.9584; 0.9615; 0.9615; 0.9642)$	0.9658
KGH	0.9625	$\overleftrightarrow{\mathcal{Tr}}(0.9592; 0.9599; 0.9650; 0.9678)$	0.9697
LTS	0.9569	$\overleftrightarrow{\mathcal{Tr}}(0.9606; 0.9583; 0.9555; 0.9535)$	0.9616
OPL	0.9652	$\overleftrightarrow{\mathcal{Tr}}(0.9520; 0.9536; 0.9768; 0.9768)$	0.9662
PGE	0.9770	$\overleftrightarrow{\mathcal{Tr}}(0.9789; 0.9789; 0.9751; 0.9732)$	0.9725
PKO	0.9597	$\overleftrightarrow{\mathcal{Tr}}(0.9530; 0.9530; 0.9666; 0.9666)$	0.9624

Source: Own elaboration.

The replacement of an accurate PV evaluation by its assessment approximated in a more accurate way, reflects the essence of the PV. If now we estimate PV with the use of TrOFN presented in Table 1 then using the Roy criterion goes down to the comparison of an imprecise OEF with the precise SPT [Łyczkowska-Hanćkowiak, 2019]. By means of (24) we then estimate the values of recommendation choice function presented in Table 3.

Tab. 3 Imprecise recommendations

	Recommendation Choice Function				
Stock Company	A^{--}	A^{-}	A^{0}	A^{+}	A^{++}
ACP	1	1	0	0	0
CPS	1	1	0	0	0
ENG	0	1	1	1	0

JSW	1	1	0	0	0
KGH	1	1	0	0	0
LTS	1	1	0	0	0
OPL	0	1	1	1	0
PGE	0	0	0	1	1
PKO	0	1	1	1	0

.Source: Own elaboration.

8. Conclusions

In the paper, it is indicated that, if the premise to formulate an investment recommendation uses the expected oriented fuzzy discount factor, then the recommendation itself is a fuzzy subset in a rating scale. This way, an investment recommendation form identical to the investment recommendation in (Piasecki, 2014) was achieved. The values of the membership function of investment recommendations should be interpreted as a degree of a chosen advice recommendation, i.e. the declared involvement of the advisor in the responsibility of taking the final investment decision. In the future it will enable to examine the impact of the discount factor orientation on the form of the investment recommendation. The topic of further research should be the determination of such successive investment strategies for which the oriented fuzzy discount factor is a basic premise to take the investment decisions.

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