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# ORDERED FUZZY NUMBERS VS FUZZY NUMBERS – THE FIRST OBSERVATIONS

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***Abstract:*** *An imprecise number is an approximation of fixed value crisp number. A commonly accepted model of imprecise number is the fuzzy number, defined as a fuzzy subset of the real line. Any ordered fuzzy number is defined as imprecise number with additional information about the location of the approximated number. The main purpose of this article is to compare the basics of ordered numbers and the theory of fuzzy numbers. Both theories differ in the definition of the unary operator "minus". For this reason, we will place focus on comparison of the arithmetic of fuzzy numbers with the arithmetic of ordered fuzzy numbers. We have shown here that, despite the identical membership functions, fuzzy numbers cannot be considered as positively oriented fuzzy numbers. For the convenience of reasoning in this article, we will limit ourselves to the case, we restrict our considerations to the case of trapezoidal fuzzy numbers. Nevertheless, all obtained conclusions may be easily generalized for case of any ordered fuzzy numbers.*

***Key words:*** *ordered fuzzy number, fuzzy number, disorientation, fuzzy arithmetic*
***JEL codes:*** *C02, C44, C63*

## 1. Introduction

The intuitive concept of ordered fuzzy number (OFN) was introduced by Kosiński and his co-workers (Kosiński et al, 2002), (Kosiński, 2006) as an extension of the concept of fuzzy number (FN) which is interpreted as imprecise approximation of real number. OFNs' usefulness follows from the fact that is interpreted as FN with additional information about the location of the approximated number. Kosiński (2006) has determined OFNs' arithmetic as an extension of results obtained by Goetschel and Voxman (1986) for FNs. OFNs' usefulness follows from the fact that is interpreted as FN with additional information about the location of the approximated number. For formal reasons, the Kosiński' theory is revised (Piasecki, 2018) in this way that new definition of OFN fully corresponds to the intuitive defining it by Kosiński. This paper fully bases on the revised OFNs' theory.

OFNs have already begun to find their use in operations research applied in decision making, economics and finance. Unfortunately, the OFN theory has one significant drawback. This drawback is the lack of formal mathematical models dedicated to OFN issues. An important goal of further formal research should be to fill these theoretical gaps.

The main aim of presented work is to present the results of our observations of the formal difference between the FN and OFN. For greater clarity of exposition, we confine our discussion to the case of trapezoidal numbers. The paper is organised as follows. Chapter 2 presents the concept of fuzzy set defined as some extension of isomorphism between crisp sets and indicator functions. Chapter 3 briefly describes the idea of trapezoidal FN. In Chapter 4, we introduce the notion of trapezoidal OFN. The same chapters describe arithmetic operations on these numbers. In Chapter 5 the authors presents results of their observations differences between FN and OFN. Finally, Chapter 6 concludes the article, summarizes the main findings of this research and proposes some generalization of obtained results.

## 2. Fuzzy sets – basic facts

The space of all declarative sentences is noted by the symbol $\mathbb{P}$. P Subjects of any cognitive-application activity are elements of a space $\mathbb{X}$. The basic tool for classifying these elements is the concept of a set. For any predicate $\varphi_A: \mathbb{X} \to \mathbb{P}$, the set $A \subset \mathbb{X}$ can be determine in the following way

$$A = \{x \epsilon \mathbb{X}: \varphi_A(x)\}. \tag{1}$$

The predicate $\varphi_A \epsilon \mathbb{P}^{\mathbb{X}}$ is called predicate of the set $A \subset \mathbb{X}$. Any set and its predicate are one-to-one linked. For unique determination of a set form, it is necessary to determine the manner in which the relationship between the actual state of affairs and the information contained in the sentence about this state is given.

The starting point for discussion on this topic is to reduce our considerations to the classical propositional calculus. The subject of the classical propositional calculus are only those declarative sentences that are true or false. Any sentences that meet these conditions are called logical sentences. The space of all logical sentences is noted by the symbol $\mathbb{P}_0 \subset \mathbb{P}$.

For each true sentence $\mathscr{p} \in \mathbb{P}_0$, we assign the truth value

$$v(\mathscr{p}) = 1. \tag{2}$$

For each false sentence $\mathscr{p} \in \mathbb{P}_0$, we assign the truth value

$$v(\mathscr{p}) = 0. \tag{3}$$

In this way, we define the function of logical evaluation $v \in \{0; 1\}^{\mathbb{P}_0}$. Any set

$$A = \{x\epsilon\mathbb{X}: \varphi_A(x),\ \varphi_A \in \mathbb{P}_0^{\mathbb{X}}\} \tag{4}$$

is a set (crisp set) described in the classic set theory. The family of all such sets is noted by the symbol $\mathcal{B}(\mathbb{X})$.

For any set $A \in \mathcal{B}(\mathbb{X})$, we determine its indicator function $\chi_A \in \{0,1\}^{\mathbb{X}} = 2^{\mathbb{X}}$ given by the identity

$$\chi_A(x) = v(\varphi_A(x)). \tag{5}$$

The indicator function value $\chi_A(x)$ is equal to the truth value of the sentence "$x \in A$".

Two-valued logic has been criticized many times. Therefore, it was extended by Łukasiewicz (1922) to multivalued logic. The subject of considerations in multivalued logic are those sentences for which the connex relation "*no less true*" is uniquely defined. The space of all sentences meeting this condition is noted by the symbol $\mathbb{P}_1 \subset \mathbb{P}$. We have $\mathbb{P}_0 \subset \mathbb{P}_1 \subset \mathbb{P}$. Fot each sequence $\mathscr{p} \in \mathbb{P}_1$, we assign the truth value $\tilde{v}(\mathscr{p})$ understood as a "*degree in which evaluated sentence is truth*". Because multivalued logic is an extension of two-valued logic, for any sentence $\mathscr{p} \in \mathbb{P}_0$ we have

$$\tilde{v}(\mathscr{p}) = v(\mathscr{p}). \tag{6}$$

In this way, we define the function of logical evaluation $\tilde{v} \in \{0; 1\}^{\mathbb{P}_1}$. Any set

$$A = \{x\epsilon\mathbb{X}: \varphi_A(x),\ \varphi_A \in \mathbb{P}_1^{\mathbb{X}}\} \tag{7}$$

is a fuzzy set intuitively introduced by Zadeh (1965). The family of all fuzzy sets is noted by the symbol $\mathcal{F}(\mathbb{X})$. For any fuzzy set $A \in \mathcal{F}(\mathbb{X})$, we determine its membership function $\mu_A \in [0,1]^{\mathbb{X}}$ given by the identity

$$\mu_A(x) = \tilde{v}(\varphi_A(x)). \tag{8}$$

The membership function value $\mu_A(x)$ is equal to the truth value of the sentence "$x \in A$".

We can to define any fuzzy set in more formal way. The spaces $\mathcal{B}(\mathbb{X})$ and $2^{\mathbb{X}}$ are isomorphic. This isomorphism is determined by the bijection $\Phi: 2^{\mathbb{X}} \to \mathcal{B}(\mathbb{X})$. The bijection $\Phi$ is increasing i.e.

$$\forall_{\chi_A,\chi_B\in 2^{\mathbb{X}}}: \quad \chi_A \leq \chi_B \Longrightarrow \Phi(\chi_A) \subset \Phi(\chi_B). \tag{9}$$

Then we have

$$\mathcal{B}(\mathbb{X}) = \Phi(2^{\mathbb{X}}). \tag{10}$$

It means that the family $\mathcal{B}(\mathbb{X})$ of all crisp sets is determined by isomorphism $\Phi$ as the image of the family $2^{\mathbb{X}}$ of all indicator function on the space $\mathbb{X}$. Because $2^{\mathbb{X}} \subset [0,1]^{\mathbb{X}}$, we can extend the isomorphism $\Phi: 2^{\mathbb{X}} \to \mathcal{B}(\mathbb{X})$ to an increasing injection $\widetilde{\Phi}$ determined on $[0,1]^{\mathbb{X}}$. For any membership function $\mu_A \in [0,1]^{\mathbb{X}}$, the element $A$ fulfilling the condition

$$A = \widetilde{\Phi}(\mu_A) \tag{11}$$

is called a fuzzy set. Then, the space $\mathcal{F}(\mathbb{X})$ is determined in the following way

$$\mathcal{F}(\mathbb{X}) = \widetilde{\Phi}([0;1]^{\mathbb{X}}). \tag{12}$$

Increasing bijection $\widetilde{\Phi}$ used above is not uniquely defined. The identity (8) shows that the unique form of the isomorphism depends on the type of multivalued logic used. On the other hand, this multivalued logic determines the set operators in $\mathcal{F}(\mathbb{X})$. In our considerations, a set operators are determined in the following way

$$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x) = \max\{\mu_A(x), \mu_B(x)\}, \tag{13}$$

$$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x) = \min\{\mu_A(x), \mu_B(x)\}, \tag{14}$$

$$\mu_{A^c}(x) = 1 - \mu_A(x). \tag{15}$$

The development trends of fuzzy set theory are restricted by Zadeh's extension principle (Zadeh, 1975a, b, c). Let us take into account fixed notion explicitly defined in the space $\mathcal{B}(\mathbb{X})$. Then any generalization of this notion can only be generalized to the space $\mathcal{F}(\mathbb{X})$ by such definition which is an extension of the definition already formulated in the space $\mathcal{B}(\mathbb{X})$.

## 3. Fuzzy number – basic facts

An imprecise number is a family of values in which each considered value belongs to it in a varying degree. A commonly accepted model of imprecise number is a FN, defined as a fuzzy subset of the family $\mathbb{R}$ of all real numbers. The most general definition of FN was given by Dubois and Prade (1978). In this paper, we restrict our considerations to the case of TrFNs defined as fuzzy subsets in the space $\mathbb{R}$ of all real numbers in the following way.

**Definition 1.** *For any nondecreasing sequence* $(a, b, c, d) \subset \mathbb{R}$, *the trapezoidal fuzzy number (TrFN) is the fuzzy subset* $\mathcal{F}(\mathbb{R}) \ni \mathcal{T} = Tr(a, b, c, d)$ *determined explicitly by its membership functions* $\mu_T \in [0,1]^{\mathbb{R}}$ *as follows*

$$\mu_T(x) = \mu_{Tr}(x|a,b,c,d) = \begin{cases} 0, & x \notin [a,d], \\ \frac{x-a}{b-a}, & x \in [a,b[, \\ 1, & x \in [b,c], \\ \frac{x-d}{c-d}, & x \in ]c,d]. \end{cases} \tag{16}$$

The space of all TrFNs is denoted by the symbol $\mathbb{F}_{Tr}$. The TrFN $Tr(a,a,a,a) = [\![a]\!]$ represents the crisp number $a \in \mathbb{R}$. Therefore, we can write $\mathbb{R} \subset \mathbb{F}_{Tr}$. For any $z \in [b,c]$, the TrFN $Tr(a,b,c,d)$ is interpreted as an imprecise number "about $z$". Understanding the phrase "about $z$" depends on the applied pragmatics of the natural language.

Let us take into account any arithmetic operation $*$ defined on $\mathbb{R}$. By $\circledast$ we denote an extension of arithmetic operation $*$ to $\mathbb{F}_{Tr}$. According to the Zadeh's Extension Principle, for any pair $(\mathcal{K},\mathcal{L}) \in (\mathbb{F}_{Tr})^2$ represented respectively by their membership functions $\mu_K, \mu_L \in [0,1]^{\mathbb{R}}$, the FN

$$\mathcal{M} = \mathcal{K} \circledast \mathcal{L} \tag{17}$$

is represented by its membership function $\mu_M \in [0,1]^{\mathbb{R}}$ given by the identity:

$$\mu_M(z) = \sup\{\min\{\mu_K(x), \mu_L(y)\} : z = x * y, (x,y) \in \mathbb{R}\}. \tag{18}$$

In line with above, the sum $\oplus$ of TrFNs and difference $\ominus$ between TrFNs are the TrFNs given as follows

$$Tr(a+e, b+f, c+g, d+h) = Tr(a,b,c,d) \oplus Tr(e,f,g,h), \tag{19}$$

$$Tr(a-h, b-g, c-f, d-e) = Tr(a,b,c,d) \ominus Tr(e,f,g,h). \tag{20}$$

In analogous way, the unary minus operator "$-$" on $\mathbb{R}$ is extended to the minus operator $\ominus$ on $\mathbb{F}_{Tr}$ by the identity

$$Tr(-h,-g,-f,-e) = [\![0]\!] \ominus Tr(e,f,g,h) = \ominus Tr(e,f,g,h). \tag{21}$$

## 4. Ordered fuzzy numbers

In this paper, we restrict our considerations to the case of TrOFN defined in the following way.

**Definition 2.** (Piasecki, 2018) *For any monotonic sequence* $(a,b,c,d) \subset \mathbb{R}$, *the trapezoidal ordered fuzzy number (TrOFN)* $\overleftrightarrow{Tr}(a,b,c,d) = \overleftrightarrow{\mathcal{T}}$ *is the pair of the orientation* $\langle a \rightarrowtail b \rangle = (a,d)$ *and fuzzy subset* $\mathcal{T} \in \mathcal{F}(\mathbb{R})$ *determined explicitly by its membership functions* $\mu_T \in [0,1]^{\mathbb{R}}$ *as follows*

$$\mu_T(x) = \mu_{Tr}(x|a,b,c,d) = \begin{cases} 0, & x \notin [a,d] \equiv [d,a], \\ \frac{x-a}{b-a}, & x \in [a,b[ \equiv ]b,a] \\ 1, & x \in [b,c] \equiv [c,b] \\ \frac{x-d}{c-d}, & x \in ]c,d] \equiv [d,c[. \end{cases} \tag{22}$$

The space of all TrOFNs is denoted by the symbol $\mathbb{K}_{Tr}$. Any TrOFN is interpreted as imprecise number with additional information about the location of the approximated number. This information is given as orientation of TrOFN. The fulfilment of the condition $a < d$ determines the positive orientation of TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$. For any $z \in [b,c]$, the positively oriented TrOFN $Tr(a,b,c,d)$ is interpreted as an imprecise number "about or slightly above $z$". The space of all positively oriented TrOFN is denoted by the symbol $\mathbb{K}_{Tr}^{+}$. The fulfilment of the condition $a > d$ determines the negative orientation of TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$. For any $z \in [c,b]$, the negatively oriented TrOFN $Tr(a,b,c,d)$ is interpreted as an imprecise number "about or slightly below $z$". The space of all negatively oriented OFN is denoted by the symbol $\mathbb{K}_{Tr}^{-}$. For the case $a = d$, TrOFN $\overleftrightarrow{Tr}(a,a,a,a) = [\![a]\!]$ represents a crisp number $a \in \mathbb{R}$, which is not oriented. Understanding the phrases "about or slightly above $z$" and "about or slightly below $z$" depend on the applied pragmatics of the natural language. Summing up, we can write

$$\mathbb{K}_{Tr} = \mathbb{K}_{Tr}^{+} \cup \mathbb{R} \cup \mathbb{K}_{Tr}^{-}. \tag{23}$$

Let us consider the TrOFNs $\overleftrightarrow{\mathcal{X}}, \overleftrightarrow{\mathcal{Y}}, \overleftrightarrow{\mathcal{W}} \in \mathbb{K}_{Tr}$ described as follows $\overleftrightarrow{\mathcal{X}} = \overleftrightarrow{Tr}(a_X, b_X, c_X, d_X)$, $\overleftrightarrow{\mathcal{Y}} = \overleftrightarrow{Tr}(a_Y, b_Y, c_Y, d_Y)$, $\overleftrightarrow{\mathcal{W}} = \overleftrightarrow{\mathcal{X}}(a_W, b_W, c_W, d_W)$. Any arithmetic operations $*$ on $\mathbb{R}$ is extended now to arithmetic operation $\boxed{*}$ on $\mathbb{K}$ by the identity

$$\overleftrightarrow{\mathcal{W}} = \overleftrightarrow{\mathcal{X}} \boxed{*} \overleftrightarrow{\mathcal{Y}} \tag{24}$$

where we have

$$\breve{a}_W = a_X * a_Y, \tag{25}$$

$$b_W = b_X * b_Y, \tag{26}$$

$$c_W = c_X * c_Y, \tag{27}$$

$$\check{d}_W = d_X * d_Y. \tag{28}$$

$$a_W = \begin{cases} \min\{\breve{a}_W, b_W\}, & (b_W < c_W) \vee (b_W = c_W \wedge \breve{a}_W \le \check{d}_W), \\ \max\{\breve{a}_W, b_W\}, & (b_W > c_W) \vee (b_W = c_W \wedge \breve{a}_W > \check{d}_W), \end{cases} \tag{29}$$

$$d_W = \begin{cases} \max\{\check{d}_W, c_W\}, & (b_W < c_W) \vee (b_W = c_W \wedge \breve{a}_W \le \check{d}_W), \\ \min\{\check{d}_W, c_W\}, & (b_W > c_W) \vee (b_W = c_W \wedge \breve{a}_W > \check{d}_W). \end{cases} \tag{30}$$

The sum $\boxplus$ of TrFNs and difference $\boxminus$ between TrFNs are the TrFNs given as follows

$$\overleftrightarrow{Tr}(a,b,c,d) \boxplus \overleftrightarrow{Tr}(p-a,q-b,r-c,s-d) =$$

$$= \begin{cases} \overleftrightarrow{Tr}(\min\{p,q\},q,r,\max\{r,s\}), & (q<r) \vee (q=r \wedge p \le s), \\ \overleftrightarrow{Tr}(\max\{p,q\},q,r,\min\{r,s\}), & (q>r) \vee (q=r \wedge p > s), \end{cases} \tag{31}$$

$$\overleftrightarrow{Tr}(a,b,c,d) \boxminus \overleftrightarrow{Tr}(a-p,b-q,c-r,d-s) =$$

$$= \begin{cases} \overleftrightarrow{Tr}(\min\{p,q\},q,r,\max\{r,s\}), & (q<r) \vee (q=r \wedge p \le s), \\ \overleftrightarrow{Tr}(\max\{p,q\},q,r,\min\{r,s\}), & (q>r) \vee (q=r \wedge p > s). \end{cases} \tag{32}$$

The unary minus operator "$-$" on $\mathbb{R}$ is extended to the minus operator $\boxed{-}$ on $\mathbb{K}_{Tr}$ by the identity

$$\overleftrightarrow{Tr}(-a,-b,-c,-d) = [\![0]\!] \boxminus \overleftrightarrow{Tr}(a,b,c,d) = \boxminus \overleftrightarrow{Tr}(a,b,c,d). \tag{33}$$

## 5. Fuzzy numbers vs ordered fuzzy numbers

Let us compare the semigroups $\langle \mathbb{F}_{Tr}, \oplus \rangle$ and $\langle \mathbb{K}_{Tr}, \boxplus \rangle$.

The identities (19) and (31) implies that the number $[\![0]\!]$ is the identity element in both these semigroups.

In (Piasecki, 2018) it is shown that the addition $\boxplus$ is not associative. It implies that semigroup $\langle \mathbb{K}_{Tr}, \boxplus \rangle$ is not group. Moreover, Kosiński (2006) has shown, that for any TrOFN $\overleftrightarrow{\mathcal{K}} = \overleftrightarrow{Tr}(a,b,c,d) \in \mathbb{K}_{Tr}$ we have

$$\overleftrightarrow{\mathcal{K}} \boxminus \overleftrightarrow{\mathcal{K}} = \overleftrightarrow{Tr}(0,0,0,0) = [\![0]\!]. \tag{34}$$

Therefore, we can cay that subtraction $\boxminus$ is inverse operator to addition $\boxplus$.

The identity (14) implies that the addition $\oplus$ is associative. On the other hand, for any TrFN $\mathcal{T} = Tr(a,b,c,d) \in \mathbb{F}_{Tr} \backslash \mathbb{R}$ we have

$$\mathcal{T} \ominus \mathcal{T} = Tr(a-d,b-c,c-b,d-a) \neq [\![0]\!]. \tag{35}$$

It shows that subtraction $\ominus$ is not inverse operator to addition $\oplus$. It proves that semigroup $\langle \mathbb{F}_{Tr}, \oplus \rangle$ is not group.

All above simple conclusions imply that the semigroups $\langle \mathbb{F}_{Tr}, \oplus \rangle$ and $\langle \mathbb{K}_{Tr}, \boxplus \rangle$ cannot be considered as homomorphic algebraic structures. Any theorems on TrFNs cannot automatically extended to the case of TrOFNs.

Nevertheless, we have that for the case $a \geq d$ the membership function of TrFN $Tr(a,b,c,d)$ is equal to the membership function of TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$. This fact implies the existence of isomorphism $\Psi: (\mathbb{K}_{Tr}^{+} \cup \mathbb{R}) \to \mathbb{F}_{Tr}$ given by the identity

$$Tr(a,b,c,d) = \Psi\left(\overleftrightarrow{Tr}(a,b,c,d)\right). \tag{36}$$

This isomorphism may be extended to the space $\mathbb{K}_{Tr}$ by disorientation map $\overline{\overline{\Psi}}: \mathbb{K}_{Tr} \to \mathbb{F}_{Tr}$

given by the identity

$$\overline{\overline{\Psi}}(\overleftrightarrow{\mathcal{K}}) = \begin{cases} \Psi(\overleftrightarrow{\mathcal{K}}), & \overleftrightarrow{\mathcal{K}} \in \mathbb{K}^+_{Tr} \cup \mathbb{R}, \\ \ominus \Psi(\boxminus \overleftrightarrow{\mathcal{K}}), & \overleftrightarrow{\mathcal{K}} \in \mathbb{K}^-_{Tr}\,. \end{cases} \tag{37}$$

The disorientation map may be equivalently defined as follows

$$\overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(a,b,c,d)\right) = Tr(\min\{a,d\}, \min\{b,c\}, \max\{b,c\}, \max\{a,d\}). \tag{38}$$

This disorientation map may be useful for some calculation with use TrOFNs. For example we can prove that:

**Theorem 1.** *For any TrOFN* $\overleftrightarrow{\mathcal{K}} \in \mathbb{K}_{Tr}$ *we have:*

$$\overline{\overline{\Psi}}(\boxminus \overleftrightarrow{\mathcal{K}}) = \ominus \overline{\overline{\Psi}}(\overleftrightarrow{\mathcal{K}}). \tag{39}$$

*Proof.* Let $\overleftrightarrow{\mathcal{K}} = \overleftrightarrow{Tr}(a,b,c,d) \in \mathbb{K}^+_{Tr}$ . Then we have $\boxminus \overleftrightarrow{\mathcal{K}} \in \mathbb{K}^-_{Tr}$. Using (38), for this case we get:

$$\overline{\overline{\Psi}}(\boxminus \overleftrightarrow{\mathcal{K}}) = \overline{\overline{\Psi}}\left(\boxminus \overleftrightarrow{Tr}(a,b,c,d)\right) = \overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(-a,-b,-c,-d)\right) = Tr(-d,-c,-b,-a),$$

$$\ominus \overline{\overline{\Psi}}(\overleftrightarrow{\mathcal{K}}) = \ominus \overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(a,b,c,d)\right) = \ominus Tr(a,b,c,d) = Tr(-d,-c,-b,-a).$$

Let $\overleftrightarrow{\mathcal{K}} = \overleftrightarrow{Tr}(a,b,c,d) \in \mathbb{K}^-_{Tr}$. Then we have $\boxminus \overleftrightarrow{\mathcal{K}} \in \mathbb{K}^+_{Tr}$. Using (38), for this case we get:

$$\overline{\overline{\Psi}}(\boxminus \overleftrightarrow{\mathcal{K}}) = \overline{\overline{\Psi}}\left(\boxminus \overleftrightarrow{Tr}(a,b,c,d)\right) = \overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(-a,-b,-c,-d)\right) = Tr(-a,-b,-c,-d),$$

$$\ominus \overline{\overline{\Psi}}(\overleftrightarrow{\mathcal{K}}) = \ominus \overline{\overline{\Psi}}\left(\overleftrightarrow{Tr}(a,b,c,d)\right) = \ominus Tr(d,c,b,a) = Tr(-a,-b,-c,-d).$$ QED

## 6. Conclusions

The study presents the results in a formal way our observations of the difference between the TrFN and TrOFN. These theses can be generalized to the OFN case in an easy way.

However, the results obtained so far make it possible to claim that:

- additive semigroup of all FNs and additive semigroup of all OFNs cannot be considered as homomorphic algebraic structures;
- any theorems on FNs cannot automatically extended to the case of OFNs.

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