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MODIGLIANIS' COEFFICIENT FOR ORIENTED FUZZY DISCOUNT FACTOR

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Abstract: *In the article, the imprecise present value is evaluated by means of a trapezoidal oriented fuzzy number. Then expected discount factor is a trapezoidal oriented fuzzy number too. The imprecise value of this factor may be used as a decision premise in creating new investment strategies. Considered strategies are built based on a comparison of an oriented fuzzy profit index and the crisp value limit. This way we obtain imprecise investment recommendation given as a fuzzy subset in the rating scale. Financial equilibrium criteria result from as a special case of this comparison. Further in the paper, the Modiglianis' Coefficient criterion is generalized for the case when expected discount factor is given as trapezoidal oriented fuzzy number. There is shown that in fuzzy case, the Modiglianis' Coefficient Criterion is also equivalent to the Sharpe's Ratio Criterion. Obtained results show that proposed theory can be used in investment applications. All theoretical considerations are illustrated by means of simple empirical case study.*

Key words: *Modiglianis' Coefficient criterion, oriented fuzzy number, fuzzy oriented discount factor*
JEL codes: *C44, C02, G10*

1. Introduction

In general, present value (PV) is equal to current equivalent of a payment available at a fixed point in time (Piasecki, 2012). PV of future cash flow is widely accepted to be an imprecise value which may be modelling by fuzzy number (FN). If PV is evaluated by FN, then the expected return rate is a fuzzy subset in the real line. This result is a theoretical foundation for investment strategies presented in (Piasecki, 2014). Moreover, in (Piasecki & Siwek, 2018) it is shown that the fuzzy expected discount factor is a better tool for appraising considered securities than the fuzzy expected return rate. Therefore, we use an expected discount factor as premise for investment making.

Ordered FN is defined by Kosiński et al (2003). For formal reason, the original Kosiński's theory is revised in (Piasecki, 2018). Let us note that if any ordered FN is determined with use the revised theory then it is called Oriented FN (OFN).

The main goal of this paper is extension of mentioned above investment strategies for the case when PV of considered security is evaluated by OFN. Then PV is additionally equipped with forecast of PV's changes. All obtained results will be applied for extension the Modiglianis' Coefficient criterion (Modigliani, 1997) to the case of PV evaluated by OFN.

The paper is organized as follows. Section 2 outlines fuzzy OFNs and their basic properties. In Section 3 PV is evaluated by trapezoidal OFNs. The oriented fuzzy expected discount factor is determined in Section 4. An upgraded model for investment recommendations dependent on oriented fuzzy expected discount factor is described in Section 5. The Modiglianis' Coefficient Criterion is extended in Section 6. Section 7 concludes the article and proposes some future research directions.

2. Ordered Fuzzy Numbers – Basic Facts

Objects of any considerations may be given as elements of a predefined space $\mathbb{X}$. The basic tool for an imprecise classification of these elements is the notion of fuzzy sets introduced by Zadeh (1967). Any fuzzy set $\mathcal{A}$ is unambiguously determined by means of its membership function $\mu_A \in [0,1]^{\mathbb{X}}$. From the point-view of multi-valued logic (Łukasiewicz, 1922/23), the value $\mu_A(x)$ is interpreted as the truth value of the sentence "$x \in \mathcal{A}$". By the symbol $\mathcal{F}(\mathbb{X})$ we denote the family of all fuzzy sets in the space $\mathbb{X}$.

Dubois & Prade (1978) have introduced fuzzy numbers (FNs) as such a fuzzy subset in the real line which may be interpreted as imprecise approximation of a real number. The ordered FNs were intuitively introduced by Kosiński et al. (2003) as an extension of the FNs concept. Ordered FNs usefulness follows from the fact that it is interpreted as FNs with additional information about the location of the approximated number. Currently, ordered FNs defined by Kosiński are often called Kosiński's numbers (Prokopowicz & Pedrycz, 2015; Prokopowicz, 2015; Piasecki, 2019). A significant drawback of Kosiński's theory is that there exist such Kosiński's numbers which, in fact, are not FNs (Kosiński 2006). For this reason, the Kosiński's theory was revised by Piasecki (2018). If an ordered FN is determined with use of the revised definition, then it is called Oriented FN (OFN). The OFN definition fully corresponds to the intuitive Kosiński's definition of ordered FNs.

In this paper, we restrict our considerations to the case of Trapezoidal OFNs (TrOFN) defined as fuzzy subsets in the space $\mathbb{R}$ of all real numbers in the following way.

Definition 1. (Piasecki, 2018) For any monotonic sequence $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) \subset \mathbb{R}$, the trapezoidal OFN (TrOFN) $\overleftrightarrow{Tr}(a,b,c,d) = \overleftrightarrow{\mathcal{T}}$ is the pair of the orientation $\langle a \rightarrowtail b \rangle = (a,d)$ and FS $\mathcal{T} \in \mathcal{F}(\mathbb{R})$ determined by membership functions $\mu_T \in [0,1]^{\mathbb{R}}$ as follows

$$\mu_T(x) = \mu_{Tr}(x|a,b,c,d) = \begin{cases} 0, & x \notin [a,d] \equiv [d,a], \\ \frac{x-a}{b-a}, & x \in [a,b[\equiv]b,a], \\ 1, & x \in [b,c] \equiv [c,b], \\ \frac{x-d}{c-d}, & x \in]c,d] \equiv [d,c[. \end{cases} \tag{1}$$

The symbol $\mathbb{K}_{Tr}$ denotes the space of all TrOFNs. Any TrOFN describes an imprecise number with additional information about the location of the approximated number. This information is given as orientation of OFN. If $a < d$ then TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$ has the positive orientation $\overrightarrow{a,d}$. For any $z \in [b,c]$, the positively oriented TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$ is a formal model of linguistic variable "about or slightly above z". If $a > d$, then OFN $\overleftrightarrow{Tr}(a,b,c,d)$ has the negative orientation $\overrightarrow{a,d}$. For any $z \in [c,b]$, the negatively oriented TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$ is a formal model of linguistic variable "about or slightly below z". Understanding the phrases "about or slightly above z" and "about or slightly below z" depends on the applied pragmatics of the natural language. If $a = d$, then TrOFN $\overleftrightarrow{Tr}(a,a,a,a) = [\![a]\!]$ describes un-oriented real number $a \in \mathbb{R}$.

On the space $\mathbb{K}_{Tr}$ we define a relation $\overleftrightarrow{\mathcal{K}}.\widetilde{GE}.\mathcal{L}$ as follows

$$\overleftrightarrow{\mathcal{K}}.\widetilde{GE}.\overleftrightarrow{\mathcal{L}} \Leftrightarrow \text{"}TrOFN\ \overleftrightarrow{\mathcal{K}}\ is\ greater\ or\ equal\ to\ TrOFN\ \overleftrightarrow{\mathcal{L}}.\text{"} \tag{2}$$

This relation is a fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}_{Tr} \times \mathbb{K}_{Tr})$ determined by its membership function $\nu_{GE} \in [0,1]^{\mathbb{K}_{Tr} \times \mathbb{K}_{Tr}}$ described in detail in (Piasecki, 2019). Due these results, for any pair $(\overleftrightarrow{Tr}(a,b,c,d), h) \in \mathbb{K}_{Tr} \times \mathbb{R}$ we have:

$$\nu_{GE}\left(\overleftrightarrow{Tr}(a,b,c,d), [\![h]\!]\right) = \begin{cases} 0, & h > \max\{a,d\}, \\ \frac{h-\max\{a,d\}}{\max\{b,c\}-\max\{a,d\}}, & \max\{a,d\} \geq h > \max\{b,c\}, \\ 1, & \max\{b,c\} \geq h, \end{cases} \tag{3}$$

$$\nu_{GE}\left([\![h]\!], \overleftrightarrow{Tr}(a,b,c,d)\right) = \begin{cases} 0, & h < \min\{a,d\}, \\ \frac{h-\min\{a,d\}}{\min\{b,c\}-\min\{a,d\}} & \min\{a,d\} \leq h < \min\{b,c\}, \\ 1 & \min\{b,c\} \leq h. \end{cases} \tag{4}$$

3. Oriented fuzzy present value

PV of the future cash flow may be imprecise. It implies that PV is described by FN. Kuchta (2000) shows the sensibility of using trapezoidal FNs as an imprecise financial arithmetic tool. Moreover, PV estimation should be supplemented by forecast of PV closest changes. For these reasons, PV is estimated by TrOFN

$$\overleftrightarrow{PV} = \overleftrightarrow{Tr}(V_s, V_f, V_l, V_e), \tag{5}$$

where the monotonic sequence $(V_s, V_f, \check{C}, V_l, V_e)$ is defined as follows

- $\check{C}$ – market price,
- $[V_s, V_e] \subset \mathbb{R}^+$ is an interval of all possible PV values,
- $[V_f, V_l] \subset [V_s, V_e]$ is an interval of all prices which do not perceptibly differ from the market price $\check{C}$.

If $V_s < V_e$, then the positively oriented PV predicts a rise in market price. If $V_s > V_e$, then the negative orientation is the forecast of a fall in market price.. Such PV is called the oriented PV (OPV).

Example 1 (Łyczkowska-Hanćkowiak, 2019): We evaluate the portfolio π composed of blocks of shares in Assecopol (ACP), ENERGA (ENG), JSW (JSW), KGHM (KGH), LOTOS (LTS), ORANGEPL (OPL) and PKOBP (PKO). Based on closing of the session on the Warsaw Stock Exchange on January 15, 2018, for each considered share we his way shares' OPVs are presented in Table 1 (Łyczkowska & Piasecki, 2018a). For each portfolio component, we determine its market price $\check{C}_s$ as initial price on 16.01.2018.

Table 1. Evaluation of portfolio π components.

Stock Company	Present Value $\overleftrightarrow{PV}_s$	Market Price $\check{C}_s$	Expected Return Rate $\bar{r}_s$	Variance σ_s^2
ACP	$\overleftrightarrow{Tr}(45.90;\ 45.90;\ 45.50;\ 45.48)$	45.70	0.0300	0.000090
CPS	$\overleftrightarrow{Tr}(22.92;\ 22.82;\ 22.82;\ 22.76)$	22.82	0.0355	0.000190
ENG	$\overleftrightarrow{Tr}(10.22;\ 10.19;\ 10.17;\ 10.14)$	10.18	0.0150	0.000020
JSW	$\overleftrightarrow{Tr}(92.24;\ 92.54;\ 92.54;\ 92.80)$	92.54	0.0400	0.000290
KGH	$\overleftrightarrow{Tr}(102.65;\ 103.05;\ 103.60;\ 103.90)$	103.33	0.0390	0.000210
LTS	$\overleftrightarrow{Tr}(56.70;\ 56.56;\ 56.40;\ 56.28)$	56.48	0.0450	0.000390
OPL	$\overleftrightarrow{Tr}(5.75;\ 5.76;\ 5.90;\ 5.90)$	5.83	0.0360	0.000280
PGE	$\overleftrightarrow{Tr}(10.39;\ 10.39;\ 10.35;\ 10.33)$	10.37	0.0235	0.000160
PKO	$\overleftrightarrow{Tr}(42.61;\ 42.61;\ 43.22;\ 43.22)$	42.91	0.0420	0.000370

We notice that the stock companies KGH, JWS, OPL and PKO are evaluated by positively oriented PV, which predicts a rise in market price. Similarly, the stock companies ACP, CPS, ENG, LTS and PGE are evaluated by negatively oriented PV, which predicts a fall in market price.

4. Oriented fuzzy discount factor

We assume that the time horizon of an investment is fixed. Then, the security considered is determined by two values: anticipated $FV = V_t$ and assessed $PV = V_0$. The basic characteristic of benefits from owning this security is the simple return rate. In practice of financial markets analysis, the uncertainty risk is usually described by probability distribution of return rate calculated for $V_0 = \check{C}$. Then the expected discount factor (EDF) $\bar{v} \in \mathbb{R}$ is given by the identity:

$$\bar{v} = \frac{1}{1+\bar{r}}\,. \tag{6}$$

Example 2 All considerations in the paper are run for the quarterly period of investment time $t = 1\ quarter$. Expected return rates of portfolio components π and their variances are presented in Table 1.

In (Łyczkowska-Hanćkowiak & Piasecki, 2018b) it is proved that if oriented EDF (OEDF) is determined by OPV (5) then it is described by TrOFN

$$\overleftrightarrow{V} = \overleftrightarrow{Tr}\left(\frac{V_s}{\check{C}} \cdot \bar{v}, \frac{V_f}{\check{C}} \cdot \bar{v},\ \frac{V_l}{\check{C}} \cdot \bar{v}, \frac{V_e}{\check{C}} \cdot \bar{v}\right). \tag{7}$$

Example 3: Using (6) and (7), we calculated quarterly EDF and OEDF for each component of the portfolio π. Obtained in this way evaluations are presented in Table 2.

The discount factor of a security described in this way is an TrOFN with the identical orientation as the oriented present value use for estimation. It is worth stressing that the maximum criterion of the expected return rate can be equivalently replaced by the minimum criterion of the EDF.

5. Investment recommendations

The investment recommendation is the counsel given by the advisors to the investor. In this paper we will consider the family of standardized advices which are applied in (Piasecki, 2014). The rating scale is given as the set $\mathbb{A} = \{A^{++},\ A^{+},\ A^{0},\ A^{-},\ A^{--}\}$, where

- A^{++} denotes the advice Buy,
- A^{+} denotes the advice Accumulate,
- A^{0} denotes the advice Hold,

- A^{-} denotes the advice Reduce,
- A^{--} denotes the advice Sell.

Let us take into account a fixed security $\check{S}$, represented by the pair $(\bar{r}_s,\ \varpi_s)$, where $\bar{r}_s$ is an expected return on $\check{S}$ and ϖ_s is a parameter characterizing the security $\check{S}$. By symbol $\mathbb{S}$ we denote the set of all considered securities. Adviser's counsel depends on the expected return. The criterion for a competent choice of advice can be presented as a comparison of the values profit $g(\bar{r}_s|\varpi_s)$ and the profitability threshold (PT) $\check{G}$, where $g(\cdot\,|\varpi_s)\colon \mathbb{R} \to \mathbb{R}$ is an increasing function of the expected return rate. The advice choice function $\Lambda\colon \mathbb{S} \times \mathbb{R} \to 2^{\mathbb{A}}$ was given in following way (Piasecki, 2014)

- $A^{++} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\varpi_s) > \check{G} \Leftrightarrow \neg g(\bar{r}_s|\varpi_s) \le \check{G},$ (8)
- $A^{+} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\varpi_s) \ge \check{G},$ (9)
- $A^{0} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\varpi_s) = \check{G} \Leftrightarrow \wedge\, g(\bar{r}_s|\varpi_s) \ge \check{G} \wedge g(\bar{r}_s|\varpi_s) \le \check{G},$ (10)
- $A^{-} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\varpi_s) \le \check{G},$ (11)
- $A^{--} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\varpi_s) < \check{G} \Leftrightarrow \neg g(\bar{r}_s|\varpi_s) \ge \check{G}.$ (12)

This way was assigned the advice subset $\Lambda(\check{S}, \check{G}) \subset \mathbb{A}$ which is interpreted as the investment recommendation given for the security.

The security $\check{S}$ may be equivalently represented by the ordered pair $(\bar{v}_s,\ \varpi_s)$, where $\bar{v}_s$ is EDF determined by (6). It is very easy to check that we have

$$g(\bar{r}_s|\varpi_s) \ge \check{G} \Leftrightarrow \bar{v}_s \le \frac{1}{1+g^{-1}(\check{G}|\varpi_s)} = H_s(\check{G}), \quad (13)$$

$$g(\bar{r}_s|\varpi_s) \le \check{G} \Leftrightarrow \bar{v}_s \ge H_s(\check{G}). \quad (14)$$

The value H_s is interpreted as a specific profitability threshold (SPT) determined for the security $\check{S}$. Then advice choice function $\Lambda\colon \mathbb{S} \times \mathbb{R} \to 2^{\mathbb{A}}$ is equivalently described in a following way

- $A^{++} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \neg \bar{v}_s \ge H_s(\check{G}),$ (15)
- $A^{+} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \le H_s(\check{G}),$ (16)
- $A^{0} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \le H_s(\check{G}) \wedge \bar{v}_s \ge H_s(\check{G}),$ (17)
- $A^{-} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \ge H_s(\check{G}),$ (18)
- $A^{--} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \neg \bar{v}_s \le H_s(\check{G}).$ (19)

We consider the case when the security $\check{S}$ is represented by the ordered pair $(\overleftrightarrow{V}_s,\ \varpi_s)$ where $\overleftrightarrow{V}_s \in \mathbb{K}_{Tr}$ is OEDF calculated with use (7). Then advice choice function $\tilde{\Lambda}(\check{S}, \check{G})$ is determined by its membership function $\lambda\left(\cdot\,\middle|\check{S},\check{G}\right):\mathbb{A}\rightarrow[0,1]$ determined in line with (15) – (19) in the following way:

- $\lambda\left(A^{++}\middle|\check{S},\check{G}\right)=1-\nu_{GE}\left(\overleftrightarrow{V_s},\llbracket H_s(\check{G})\rrbracket\right)$, (20)
- $\lambda\left(A^{+}\middle|\check{S},\check{G}\right)=\nu_{GE}\left(\llbracket H_s(\check{G})\rrbracket,\overleftrightarrow{V_s}\right)$, (21)
- $\lambda\left(A^{0}\middle|\check{S},\check{G}\right)=\min\left\{\nu_{GE}\left(\llbracket H_s(\check{G})\rrbracket,\overleftrightarrow{V_s}\right),\ \nu_{GE}\left(\overleftrightarrow{V_s},\llbracket H_s(\check{G})\rrbracket\right)\right\}$, (22)
- $\lambda\left(A^{-}\middle|\check{S},\check{G}\right)=\nu_{GE}\left(\overleftrightarrow{V_s},\llbracket H_s(\check{G})\rrbracket\right)$, (23)
- $\lambda\left(A^{--}\middle|\check{S},\check{G}\right)=1-\nu_{GE}\left(\llbracket H_s(\check{G})\rrbracket,\overleftrightarrow{V_s}\right)$. (24)

From the point-view of decision-making, the value $\lambda\left(A\middle|\check{S},\check{G}\right)$ is interpreted as a degree of recommendation support $A\in\mathbb{A}$, i.e. a declared share of the advisor in the responsibility in case of final decision-making according to the advice $A\in\mathbb{A}$. In the described situation the investment recommendation $\tilde{\Lambda}\left(\check{S},\check{G}\right)$ is the fuzzy subset in the rating scale $\mathbb{A}$.

6. The Modiglianis' Coefficient

Modiglianis' Coefficient is one of the criteria of risk management. In crisp case, the Modiglianis' Coefficient Criterion is equivalent to the Sharpe's Ratio Criterion. In this model of financial equilibrium, the compared values are the expected return on a security and the expected return on the market portfolio. Modiglianis' profit coefficient estimates bonus over market profits. Modiglianis' limit value is equal zero. We assume that there exists the risk-free bond instrument represented by the pair $(r_0,0)$ and the market portfolio represented by the pair $(r_M,{\sigma_M}^2)$.

Example 4. We focus on the Warsaw Stock Exchange. We consider financial market with risk-free bound instrument determined as quarterly treasure bounds with return rate $r_0=0.0075$. The market portfolio is represented by portfolio determining stock exchange index WIG20 which is represented by the pair $(r_M,{\sigma_M}^2)=(0.0200,0.000025)$.

If the security $\check{S}$ is represented by the pair $(\bar{r}_s,\sigma_S^2)$, then Modigliani (1997) defines the profit index $g(\cdot\,|\sigma_s):\mathbb{R}\rightarrow\mathbb{R}$ and the PT $\check{G}$ as follows:

$$g(\bar{r}_s|\sigma_s)=r_0-r_M+\frac{r_s-r_0}{\sigma_s}\cdot\sigma_M, \tag{25}$$

$$\check{G}=0. \tag{26}$$

For this case, we calculate SPT H_s in following manner

$$H_s=\frac{\sigma_M}{\sigma_s\cdot(r_M-r_0)+\sigma_M\cdot(r_0+1)}. \tag{27}$$

We see that in fuzzy case, the Modiglianis' Coefficient Criterion is also equivalent to the Sharpe's Ratio Criterion described in (Łyczkowska-Hanćkowiak, 2019).

Example 5: Using (27), we calculate SPT for each component of the portfolio π. Evaluations obtained in this way are presented in Table 2.

The replacement of an accurate PV evaluation by its assessment approximated in a more accurate way, reflects the essence of the PV. If now we estimate PV with the use of TrOFN presented in Table 1 then using the Modiglianis' Coefficient criterion goes down to the comparison of an imprecise OEDF with the precise SPT. By means of (20) – (24) we estimate the values of recommendation choice function presented in Table 3.

Table 2 Expected discount factors of portfolio π components.

Stock Company	EDF $\overline{v}_s$	OEDF $\overleftrightarrow{V}_s$	SPT H_s
ACP	0.9709	$\overleftrightarrow{Tr}(0.9751; 0.9751; 0.9666; 0.9662)$	0.9697
CPS	0.9657	$\overleftrightarrow{Tr}(0.9699; 0.9657; 0.9657; 0.9632)$	0.9597
ENG	0.9852	$\overleftrightarrow{Tr}(0.9891; 0.9862; 0.9842; 0.9813)$	0.9816
JSW	0.9615	$\overleftrightarrow{Tr}(0.9584; 0.9615; 0.9615; 0.9642)$	0.9524
KGH	0.9625	$\overleftrightarrow{Tr}(0.9592; 0.9599; 0.9650; 0.9678)$	0.9581
LTS	0.9569	$\overleftrightarrow{Tr}(0.9606; 0.9583; 0.9555; 0.9535)$	0.9461
OPL	0.9652	$\overleftrightarrow{Tr}(0.9520; 0.9536; 0.9768; 0.9768)$	0.9531
PGE	0.9770	$\overleftrightarrow{Tr}(0.9789; 0.9789; 0.9751; 0.9732)$	0.9622
PKO	0.9597	$\overleftrightarrow{Tr}(0.9530; 0.9530; 0.9666; 0.9666)$	0.9474

Table 3 Imprecise recommendations.

	Investment Recommendation				
Stock Company	A^{--}	A^{-}	A^{0}	A^{+}	A^{++}
ACP	0	1	1	1	0
CPS	0	1	1	1	0
ENG	0	1	1	1	0
JSW	0	1	1	1	0
KGH	0	1	1	1	0
LTS	0	1	1	1	0
OPL	0	1	1	1	0
PGE	0	1	1	1	0
PKO	0	1	1	1	0

These are ambiguous recommendations. It means that even the use of precise premises does not guarantee obtaining precise recommendations.

7. Conclusions

Obtained results may be applied in behavioural finance theory as a normative model for investment's decisions. The results may also provide theoretical foundations for constructing an investment decision support system.

The next stage of research should be to undertake studies on the imprecision of recommendations obtained as described in the article. Obtained results may well be a starting point for a future investigation of the impact of the PV's imprecision and orientation on imprecision of investment recommendation.

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