NEW METHOD OF APPROXIMATION ANY ORIENTED FUZZY NUMBER BY TRAPEZOIDAL ONE

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Abstract: *In this paper the model of imprecise quantity information is an oriented fuzzy number (OFN). The OFN ambiguity is evaluated by means of ambiguity index determined as an extension of energy measure. The OFN indistinctness is evaluated by indistinctness index determined as an extension of Czogała-Gottwald-Pedrycz entropy measure. We discuss the problem of approximation of an arbitrary OFN by the nearest trapezoidal OFN (TrOFN). In this way, we can simplify arithmetical operations on the OFNs linear space. The set of feasible TrOFNs is limited by the combination of following conditions: invariance of OFN ambiguity, invariance of OFN indistinctness and invariance of OFN support. The main purpose of our study is presentation some new methods of approximation any OFN by trapezoidal TrOFN. In new approximation tasks, the new criterion of indistinctness invariance is based on Czogała-Gottwald-Pedrycz entropy measure. The obtained results are needed for management of financial portfolio under imprecision risk.*

Key words: *oriented fuzzy numbers, approximation, information ambiguity, information indistinctness*
JEL codes: *C02, C44, G11, G40*

1. Introduction

Ordered fuzzy numbers are defined by Kosiński et al (2002) who in this way were going to introduce a fuzzy number supplemented by orientation. For some formal reason (Kosiński, 2006), the original Kosiński's theory was revised in (Piasecki, 2018). If ordered fuzzy number is linked to the revised theory, then it is called Oriented Fuzzy Number (OFN).

OFNs have already begun to find their use in operations research applied in decision making, economics and finance [7-31]. All discussed applications are associated with the linear space of OFNs. However, a big inconvenience appears when using that space. Addition of an arbitrary OFNs is a very complicated operation of a big formal complexity. On the other hand, addition of trapezoidal OFNs (TrOFNs) is reduced to a simple addition of four-dimensional vector of real numbers. This is a premise to present a substitution of OFNs by TrOFNs. The approximation problem of OFNs by TrOFNs is discussed in (Piasecki, Łyczkowska-Hanćkowiak, 2018). There the family of all feasible TrOFNs is limited by the combination of following conditions: invariance of OFN ambiguity, invariance of OFN indistinctness and invariance of OFN support. In (Piasecki, Łyczkowska-Hanćkowiak, 2018), the criterion of indistinctness invariance is based on Kosko's (1986) entropy measure. In [Łyczkowska-Hanćkowiak, Piasecki, 2018 c, d; Piasecki, Siwek, 2018a, b], it is shown that Kosko's entropy measure is not convenient for portfolio analysis. On the other hand, the entropy measure proposed by Czogała, Gottwald and Pedrycz (1981) has such usefulness. Therefore, we propose to evaluate indistinctness of arbitrary OFN by indistinctness index determined as an extension of Czogała–Gottwald–Pedrycz entropy measure.

The main aim of this paper is to present such approximation methods of any OFN by TrOFNs that they are linked to the criterion of indistinctness invariance based on indistinctness index. As a result, authors intend to indicate the recommended approximation methods.

The paper is organised as follows. Section 2 presents the basic concept of OFNs. Section 3 briefly discusses the idea of imprecision (Klir, 1993). The same chapter describes ambiguity index and indistinctness index as a tool for evaluation of OFN imprecision. In Section 4 the authors remind 4 methods of approximation methods of any OFN by a trapezoidal OFN. Moreover, there are discussed 2 new methods of approximation. Section 5 recommends the most faithful approximation methods and shows the future direction of research.

2 Oriented Fuzzy Numbers – Basic Facts

Objects of any considerations may be given as elements of a predefined space $\mathbb{X}$. The basic tool for an imprecise classification of these elements is the notion of fuzzy sets introduced by Zadeh (1965). Any fuzzy set $\mathcal{A}$ is unambiguously determined by means of its membership function $\mu_A \in [0,1]^{\mathbb{X}}$. From the point-view of multi-valued logic, the value $\mu_A(x)$ is interpreted as the truth value of the sentence "$x \in \mathcal{A}$". By the symbol $\mathcal{F}(\mathbb{X})$ we denote the family of all fuzzy sets in the space $\mathbb{X}$.

Dubois and Prade (1978) have introduced fuzzy numbers (FNs) as such a fuzzy subset in the real line which may be interpreted as imprecise approximation of a real number. The ordered FNs were intuitively introduced by Kosiński et al. (2002) as an extension of the FNs concept. Ordered FNs usefulness follows from the fact that it is interpreted as FNs with additional information about the location of the approximated number. Currently, ordered FNs defined by Kosiński are often called Kosiński's numbers (Prokopowicz, Pedrycz, 2015; Piasecki, 2019). A significant drawback of Kosiński's theory is that there exist such Kosiński's numbers which, in fact, are not FNs (Kosiński, 2006). For this reason, the Kosiński's theory was revised by Piasecki (2018). If an ordered FN is determined with use of the revised definition, then it is called Oriented FN (OFN).

Definition 1: (Piasecki, 2018) For any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$ OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is defined as the pair of orientation $\overrightarrow{a, d} = (a, d)$ and FN $\mathcal{L} \in \mathbb{F}(\mathbb{R})$ determined by its membership function $\mu_{\mathcal{L}}(\cdot\,|a, b, c, d, S_L, E_L) \in [0; 1]^{\mathbb{R}}$ given by the identity

$$\mu_{\mathcal{L}}(x|a, b, c, d, S_L, E_L) = \begin{cases} 0, & x \notin [a, d] = [d, a], \\ S_L(x), & x \in [a, b] = [b, a], \\ 1, & x \in [b, c] = [c, b], \\ E_L(x), & x \in [c, d] = [d, c] \end{cases}, \tag{1}$$

where the starting-function $S_L \in [0; 1]^{[a,b]}$ and the ending-function $E_L \in [0; 1]^{[c,d]}$ are upper semi-continuous monotonic functions satisfying the conditions

$$S_L(b) = E_L(c) = 1. \tag{2}$$

$$\forall_{x \in]a,d[} \quad \mu_{\mathcal{L}}(x|a, b, c, d, S_L, E_L) > 0\ .\square \tag{3}$$

The space of all OFN is denoted by the symbol $\mathbb{K}$. Any OFN describes an imprecise number with additional information about the location of the approximated number. This information is given as orientation of OFN. If $a < d$ then OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ has the positive orientation $\overrightarrow{a, d}$. For any $z \in [b, c]$, the positively oriented OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is a formal model of linguistic variable "about or slightly above z". If $a > d$, then OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$

has the negative orientation $\overrightarrow{a, d}$. For any $z \in [c, b]$, the negatively oriented OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is a formal model of linguistic variable "about or slightly below z". Understanding the phrases "about or slightly above z" and "about or slightly below z" depends on the applied pragmatics of the natural language. If $a = d$, then OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L) = [\![a]\!]$ describes un-oriented real number $a \in \mathbb{R}$.

The addition of OFN also is defined in (Piasecki, 2018). This addition has a high level of formal complexity (Piasecki, Łyczkowska-Hanćkowiak, 2018). Due to that, in many practical applications researchers limit the use of OFN only to a form presented below.

Definition 2. (Piasecki, 2018) For any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$ the trapezoidal OFN (TrOFN) $\overleftrightarrow{Tr}(a, b, c, d)$ is defined as the OFN determined by its membership function $\mu_{\overleftrightarrow{Tr}}(\cdot\,|a, b, c, d) \in [0; 1]^{\mathbb{R}}$ given by the identity

$$\mu_{\overleftrightarrow{Tr}}(x|a, b, c, d) = \begin{cases} 0, & x \notin [a, d] = [d, a], \\ \frac{x-a}{b-a}, & x \in [a, b[\; = \;]b, a], \\ 1, & x \in [b, c] = [c, b], \\ \frac{x-d}{c-d}, & x \in \;]c, d] = [d, c[\end{cases} \tag{4}$$

The symbol $\mathbb{K}_{Tr}$ denotes the space of all TrOFNs.The postulate to constrain to TrOFN use when constructing models of real objects not always is possible to satisfy. A behavioural present value (Piasecki, 2011; Łyczkowska-Hanćkowiak, 2018), described by such OFN that cannot be TrOFN, is an example. The attempts to use that concept in portfolio analysis bear many problems. This observation lead to a postulate to approximate all OFNs by TrOFNs while modelling all real objects. Some solutions of this task are presented in (Piasecki, Łyczkowska-Hanćkowiak, 2018). Next approximation tasks will be formulated in Chapter 4 of the paper.

3. Evaluation of Imprecision

After Klir (1993) we understand imprecision as a superposition of ambiguity and indistinctness of information. Ambiguity can be interpreted as a lack of a clear recommendation between one alternative among various others. Indistinctness is understood as a lack of explicit distinction between recommended and not recommended alternatives.

Any OFN is a particular kind of imprecision information. An increase in information imprecision reduces suitability of this information. Therefore, it is logical to consider the problem of imprecision assessment.

The ambiguity index $\alpha \in \mathbb{R}^{\mathbb{K}}$ assets the ambiguity of any OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ as follows

$$a\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right)=\int_a^d \mu_{\mathcal{L}}(x|a,b,c,d,S_L,E_L)dx. \tag{5}$$

For any negatively oriented OFN its ambiguity index is negative and for any positively oriented OFN its ambiguity index is positive. Moreover, it is obvious that for any OFN

$$d\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right)=\left|a\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right)\right|. \tag{6}$$

It all means that ambiguity index stores the information on energy measure and additionally also about the orientation of assessed OFN.

Example 1: For OFN

$$\mu_V(x)=\begin{cases}0, & x\notin[15;5],\\ S_V(x), & x\in[15;11],\\ 1, & x\in[11;6],\\ E_V(x), & x\in[6;5],\end{cases}=\begin{cases}0, & x\notin[15;5],\\ \frac{3x-45}{x-23}, & x\in[15;11],\\ 1, & x\in[11;6],\\ \frac{3x-15}{x-3}, & x\in[6;5]\ .\end{cases} \tag{7}$$

its ambiguity index is calculated as follows

$$a(\overleftrightarrow{\mathcal{V}})=\int_{15}^{11}\frac{3x-45}{x-23}dx+\int_{11}^{6}dx+\int_{6}^{5}\frac{3x-15}{x-3}dx\approx-7.8362,\ \square$$

The right tool for measuring the indistinctness is the entropy measure, proposed also by de Luca and Termini (19 and modified by Piasecki (2017). The most widely kind of entropy measure is described by Kosko (1986). On the other hand, in (Łyczkowska-Hanćkowiak, Piasecki, 2018; Piasecki, Siwek, 2018a, b), it is shown that Kosko' entropy measure is not convenient for portfolio analysis. Therefore, we propose to evaluate indistinctness of arbitrary OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ by indistinctness index $g\in\mathbb{R}^{\mathbb{K}}$ determined by the integral

$$g\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right)=\int_a^d \min\{\mu_{\mathcal{L}}(x|a,b,c,d,S_L,E_L),1-\mu_{\mathcal{L}}(x|a,b,c,d,S_L,E_L)\}\,dx \tag{8}$$

For any negatively oriented OFN its indistinctness index is negative and for any positively oriented OFN its indistinctness index is positive. Moreover, the identity

$$e\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right)=\left|g\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right)\right|. \tag{9}$$

determines the Czogała–Gottwald–Pedrycz entropy measure. It all means that indistinctness index stores the information on entropy measure and additionally also about the orientation of assessed OFN. The notions of ambiguity index and of indistinctness one give new perspectives for imprecision management.

Example 2: For OFN $\overleftrightarrow{\mathcal{V}}$ determined by (7), its indistinctness index is calculated as follows

$$g(\overleftrightarrow{\mathcal{V}})=\int_{15}^{5}\min\{\mu_V(x),1-\mu_V(x)\}dx=$$

$$= \int_{15}^{\frac{67}{5}} \frac{3x-45}{x-23} dx + \int_{\frac{67}{5}}^{11} \left(1 - \frac{3x-45}{x-23}\right) dx + \int_{11}^{6} 0 dx + \int_{6}^{\frac{46}{5}} \left(1 - \frac{3x-30}{x-14}\right) dx + \int_{\frac{46}{5}}^{5} \frac{3x-30}{x-14} dx \approx -4.9003. \ \square$$

For any TrOFN $\overleftrightarrow{Tr}(a,b,c,d)$, its ambiguity index and indistinctness one are determined basing on the following relation

$$a\left(\overleftrightarrow{Tr}(a,b,c,d)\right) = \frac{1}{2} \cdot (d + c - b - a), \tag{10}$$

$$g\left(\overleftrightarrow{Tr}(a,b,c,d)\right) = \frac{1}{2} \cdot (d - c + b - a)\ . \tag{11}$$

4. Approximation Problem

To estimate the distance between any pair of OFNs we introduce a pseudo-metrics $\delta: \mathbb{K}^2 \longrightarrow \mathbb{R}_0^+$ determined by a following identity

$$\delta\left(\overleftrightarrow{\mathcal{L}}\left(a_1, b_1, c_1, d_1, S_L^{(1)}, E_L^{(1)}\right), \overleftrightarrow{\mathcal{L}}\left(a_2, b_2, c_2, d_2, S_L^{(2)}, E_L^{(2)}\right)\right) =$$

$$= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2}. \tag{12}$$

In this section we will considered the approximation problem of an arbitrary OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ by the nearest TrOFN $\overleftrightarrow{Tr}(p,q,r,s)$. Hence, the main criterion of approximation is to determine such TrOFN $\overleftrightarrow{Tr}(p_0,q_0,r_0,s_0)$, that will satisfy the following condition

$$\delta\left(\overleftrightarrow{Tr}(p_0,q_0,r_0,s_0), \overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right) =$$

$$= \min\left\{\delta\left(\overleftrightarrow{Tr}(p,q,r,s), \overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right): \overleftrightarrow{Tr}(p,q,r,s) \in \mathbb{K}\ \right\}. \tag{13}$$

which is equivalent to a problem

$$\Phi(p_0,q_0,r_0,s_0|a,b,c,d) = \min\{\Phi(p,q,r,s|a,b,c,d): (p,q,r,s) \in \mathbb{R}^4\} \tag{14}$$

of minimization of the objective function $\Phi(\cdot\ |a,b,c,d): \mathbb{R}^4 \longrightarrow \mathbb{R}_0^+$ given by an identity

$$\Phi(p,q,r,s|a,b,c,d) = (p-a)^2 + (q-b)^2 + (r-c)^2 + (s-d)^2. \tag{15}$$

When processing imprecise values, we use OFN only to follow the influence of imprecision on the quality of obtained information. Due to in our approximation problem we can impose the requirements of estimating such an approximated value which retains the imprecision estimates of an approximated value. Using (5) and (8), we can determine the estimates of ambiguity and indistinctness of the approximated value

$$A = a\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right), \tag{16}$$

$$G = g\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right). \tag{17}$$

In that case, the conditions of imprecision we denote as

$$a\left(\overleftrightarrow{Tr}(p,q,r,s)\right) = A, \tag{18}$$

$$g\left(\overleftrightarrow{Tr}(p,q,r,s)\right) = G. \tag{19}$$

Juxtaposing the identity (10) and the condition (18) implies a linear equation describing invariance of ambiguity

$$s + r - q - p = 2 \cdot A. \tag{20}$$

Pairing the identity (11) and the condition (19) implies a linear equation describing invariance of indistinctness

$$s - r + q - p = 2 \cdot G\ . \tag{21}$$

For any OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$, the interval $[a,d]$ is the smallest closed subset containing all possible values represented by this OFN. Hence Invariant Criterion

$$[a,d] = [p,s] \tag{22}$$

can be treated as yet another constraint imposed on approximation method. This Criterion is represented by the equation system:

$$\begin{cases} p = a, \\ s = d. \end{cases} \tag{23}$$

Each TrOFN feasible in the approximation task will be called a feasible TrOFN. Any approximation problem will be distinguished by a chosen combination of restrictions (18), (19) and (22) limiting the set of feasible TrOFN. Each of approximation problems is called XYZ-approximation, where the prefix XYZ is an acronym identifying the approximation problem. The solution of XYZ-approximation of any OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ is denoted by a symbol $\overleftrightarrow{Tr}_{XYZ}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right)$. At the beginning, we present approximation tasks introduced in (Piasecki, Łyczkowska-Hanćkowiak, 2018).

Initially, we will consider the approximation problem with no constraints imposed on the set of feasible TrOFN. This approximation problem is called UC-approximation. UC-approximation is determined just by the objective function (15). In the face of a lack of any restricting equations the solution of UC-approximation problem of OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ is TrOFN

$$\overleftrightarrow{Tr}_{UC}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right) = \overleftrightarrow{Tr}(\varpi_{UC}) = \overleftrightarrow{Tr}(a,b,c,d). \tag{24}$$

Example 3: For OFN $\overleftrightarrow{\mathcal{V}} = \overleftrightarrow{\mathcal{L}}(15,11,6,5,S_V,E_V)$ determined by (7), we have

$$\overleftrightarrow{Tr}_{UC}(\overleftrightarrow{\mathcal{V}}) = \overleftrightarrow{Tr}(15,11,6,5).$$

The values $a\left(\overleftrightarrow{Tr}_{UC}(\overleftrightarrow{\mathcal{V}})\right)$ and $g\left(\overleftrightarrow{Tr}_{UC}(\overleftrightarrow{\mathcal{V}})\right)$ are presented in Table 1. □

In CA-approximation problem, each feasible TrOFN satisfies the condition (18). In (Piasecki, Łyczkowska –Hanćkowiak, 2018) it is shown that the sequence

$$\varpi_{CA} = \left(\frac{3\cdot a-b+c+d-2\cdot A}{4}, \frac{-a+3\cdot b+c+d-2\cdot A}{4}, \frac{a+b+3\cdot c-d+2\cdot A}{4}, \frac{a+b-c+3\cdot d+2\cdot A}{4}\right) \quad (25)$$

is always monotonic. Then the solution of CA-approximation task is given as follows

$$\overleftrightarrow{Tr}_{CA}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right) = \overleftrightarrow{Tr}(\varpi_{CA}). \quad (26)$$

Example 4: For OFN $\overleftrightarrow{\mathcal{V}} = \overleftrightarrow{\mathcal{L}}(15,11,6,5,S_V,E_V)$ determined by (7), we have

$$\overleftrightarrow{Tr}_{CA}(\overleftrightarrow{\mathcal{V}}) = \overleftrightarrow{Tr}(15.1681, 11.1681, 5.8319, 4.8319).$$

The values $a\left(\overleftrightarrow{Tr}_{CA}(\overleftrightarrow{\mathcal{V}})\right)$ and $g\left(\overleftrightarrow{Tr}_{CA}(\overleftrightarrow{\mathcal{V}})\right)$ are presented in Table 1□

In IC-approximation problem, each feasible TrOFN satisfies the condition (22). In (Piasecki, Łyczkowska–Hanćkowiak, 2018) it is shown that the solution of this approximation task is given, as follows

$$\overleftrightarrow{Tr}_{IC}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right) = \overleftrightarrow{Tr}(\varpi_{UC}) = \overleftrightarrow{Tr}_{UC}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right). \quad (27)$$

It means that UC-approximation and IC-approximation problems always have an identical solution. Therefore, there is no need to consider IC-approximation problem.

In ICCA-approximation problem, each feasible TrOFN satisfies the condition (18) and (22). In (Piasecki, Łyczkowska–Hanćkowiak, 2018) it is shown that the solution of ICCA-approximation task exists iff the sequence

$$\varpi_{ICCA} = \left(a, \frac{-a+b+c+d}{2} - A, \frac{a+b+c-d}{2} + A, d\right) \quad (28)$$

is monotonic. Then this solution is given as follows

$$\overleftrightarrow{Tr}_{ICCA}\left(\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)\right) = \overleftrightarrow{Tr}(\varpi_{ICCA}). \quad (29)$$

Moreover, there it is shown that exists such OFNs, for which sequence ϖ_{ICCA} is not monotonic.

Example 5: For OFN $\overleftrightarrow{\mathcal{V}} = \overleftrightarrow{\mathcal{L}}(15,11,6,5,S_V,E_V)$ determined by (7), we have

$$\overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{V}}) = \overleftrightarrow{Tr}(15, 11.3362, 5.6638, 5).$$

The values $a\left(\overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{V}})\right)$ and $g\left(\overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{V}})\right)$ are presented in Table 1□

The Constant Imprecision (CI) is understand as combination of invariant ambiguity and invariant indistinctness. Therefore, in (Piasecki, Łyczkowska –Hanćkowiak, 2018), the CI-approximation is determined as CA-approximation with additional criterion of invariant Kosko's entropy measure. In this paper, we introduce the modified CI-approximation marked by mCI-approximation acronym. In mCI-approximation problem, each feasible TrOFN satisfies the conditions (18) and (19). Then the coordinates of all feasible TrOFN $\overleftrightarrow{Tr}(p,q,r,s)$ can be represented as a general solution of the equations system (20) and (21). This solution is given in the form

$$\begin{cases} p = x \\ q = y \\ r = y + A - G \\ s = x + A + G \end{cases} \qquad x, y \in \mathbb{R}. \tag{30}$$

In order to solve the task of mCI-approximation of OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$, in the first step we find the minimum of function

$$\varphi(x, y) = \Phi(x, y, y + A - G, x + A + G \ |a, b, c, d). \tag{31}$$

This minimum is reached at the point (x_0, y_0) coordinates

$$\begin{cases} x_0 = \frac{a+d-A-G}{2} \\ y_0 = \frac{b+c-A+G}{2} \end{cases}. \tag{32}$$

Substituting (32) into (30) we obtain the particular solution

$$\varpi_{mCI} = \left(\frac{a+d-A-G}{2}, \frac{b+c-A+G}{2}, \frac{b+c+A-G}{2}, \frac{a+d+A+G}{2}\right) \tag{35}$$

of the equations system (20) and (21). The solution of mCI-approximation task exists iff the sequence ϖ_{mCI} is monotonic. Then this solution is given as follows

$$\overleftrightarrow{Tr}_{mCI}\left(\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)\right) = \overleftrightarrow{Tr}(\varpi_{mCI}). \tag{36}$$

Example 6: For OFN $\overleftrightarrow{\mathcal{V}} = \overleftrightarrow{\mathcal{L}}(15{,}11{,}6{,}5, S_V, E_V)$ determined by (7), we have

$$\overleftrightarrow{Tr}_{mCI}(\overleftrightarrow{\mathcal{V}}) = \overleftrightarrow{Tr}(16{,}7750; 15{,}2665; 13{,}7336; 14{,}2250).$$

The values $a\left(\overleftrightarrow{Tr}_{mCI}(\overleftrightarrow{\mathcal{V}})\right)$ and $g\left(\overleftrightarrow{Tr}_{mCI}(\overleftrightarrow{\mathcal{V}})\right)$ are presented in Table 1. □

The Invariant Criterion with Constant Imprecision (ICCI) is understand as combination of Invariant Criterion together with invariant ambiguity and invariant indistinctness. Therefore, in (Piasecki, Łyczkowska –Hanćkowiak, 2018), the ICCI-approximation is determined as ICCA-approximation with additional criterion of invariant Kosko's entropy measure. In this paper, we introduce the modified ICCI-approximation marked by mICCI-approximation acronym. In mICCI-approximation problem, each feasible TrOFN satisfies the conditions (18), (19) and (22). Then the coordinates of all feasible TrOFN $\overleftrightarrow{Tr}(p, q, r, s)$ can be represented as a general solution of the equations system (20), (21) and (23) which is equivalent to following equations system

$$\begin{cases} r - q = 2 \cdot A + a - d \\ -r + q = 2 \cdot G + a - d \end{cases}. \tag{37}$$

According to Cramer's formulas, this system is consistent iff

$$A + G = d - a. \tag{38}$$

On the other hand, we have:

Theorem 1: Any OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ satisfies the condition (41) iff its membership function fulfils the condition

$$\forall_{x\in[a,d]}:\ \mu_{\mathcal{L}}(x) = \ \mu_{\mathcal{L}}(x|a,b,c,d,S_L,E_L)\ \geq \frac{1}{2}. \quad \square \tag{39}$$

Proof: If the condition (39) is satisfied then we have

$$A+G = \int_a^d \mu_{\mathcal{L}}(x)dx + \int_a^d min\{\mu_{\mathcal{L}}(x), 1-\mu_{\mathcal{L}}(x)\}\,dx=$$

$$= \int_a^d \mu_{\mathcal{L}}(x)dx + \int_a^d 1-\mu_{\mathcal{L}}(x)dx = \int_a^d dx = d-a.$$

Let us assume that the condition (38) is fulfilled. According to (2), for any OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ there exists such interval $[e,f]\subset[a,d]$, that the orientations $\overrightarrow{e,f}$ and $\overrightarrow{a,d}$ are identical and the condition

$$\forall_{x\in[e,f]}:\ \mu_{\mathcal{L}}(x) = \ \mu_{\mathcal{L}}(x|a,b,c,d,S_L,E_L)\ \geq \frac{1}{2}$$

is fulfilled. Let us suppose now that $[e,f]\neq[a,d]$. Then we have

$$A+G = \int_a^d \mu_{\mathcal{L}}(x)dx + \int_a^d min\{\mu_{\mathcal{L}}(x), 1-\mu_{\mathcal{L}}(x)\}\,dx =$$

$$\int_a^e \mu_{\mathcal{L}}(x) + \min\{\mu_{\mathcal{L}}(x), 1-\mu_{\mathcal{L}}(x)\}\,dx + \int_e^f \mu_{\mathcal{L}}(x) + \min\{\mu_{\mathcal{L}}(x), 1-\mu_{\mathcal{L}}(x)\}\,dx +$$

$$\int_f^d \mu_{\mathcal{L}}(x) + \min\{\mu_{\mathcal{L}}(x), 1-\mu_{\mathcal{L}}(x)\}\,dx = 2\cdot\int_a^e \mu_{\mathcal{L}}(x)dx + (f-e) + 2\cdot\int_f^d \mu_{\mathcal{L}}(x)dx = Y\ .$$

If OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ positively oriented then we have

$$Y < (f-e) + 2\cdot\int_{[a,e]\cup[f,d]}\frac{1}{2}dx = (e-a)+(f-e)+(d-f) = d-a\ ,$$

what is contrary to (38). If OFN $\overleftrightarrow{\mathcal{L}}(a,b,c,d,S_L,E_L)$ is negatively oriented then we get

$$Y > (f-e) + 2\cdot\int_{[a,e]\cup[f,d]}\frac{1}{2}dx = (e-a)+(f-e)+(d-f) = d-a\ ,$$

which is also contrary to (38). All this proves $[e,f]=[a,d]$, which concludes the proof of the theorem. ■

We can suppose that condition (39) significantly limits the area of application of mICCI-approximation.

Example 7: The condition (39) is not fulfilled by OFN $\overleftrightarrow{\mathcal{V}}$ determined by (7). This condition is satisfied by OFN $\overleftrightarrow{\mathcal{Y}} = \overleftrightarrow{\mathcal{L}}(3;2;1;0;S_Y;E_Y)$ determined by its membership function

$$\mu_Y(x) = \begin{Bmatrix} \begin{cases} 0, & x\notin[3;0], \\ S_Y(x), & x\in[3;2], \\ 1, & x\in[2;1], \\ E_Y(x), & x\in[1;0], \end{cases} \end{Bmatrix} = \begin{cases} 0, & x\notin[3;0], \\ -\frac{1}{2}\cdot x^2 + 2\cdot x - 1, & x\in[3;2], \\ 1, & x\in[2;1], \\ -\frac{1}{2}\cdot x^2 + x + \frac{1}{2}, & x\in[1;0]\ . \end{cases} \tag{40}$$

The values $a(\mathcal{Y})$ and $g(\mathcal{Y})$ are presented in Table 1. □

If the condition (39) is fulfilled, then the general solution of equations system (37) is given as follows

$$\begin{cases} q = x \\ r = x + 2 \cdot A + a - d \end{cases}, \qquad x, y \in \mathbb{R}. \tag{41}$$

In order to solve the task of mICCI-approximation of OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$, in the first step we find the minimum of function

$$\varphi(x) = \Phi(a, x, x + 2 \cdot A + a - d, d \ |a, b, c, d). \tag{42}$$

This minimum is reached at the point

$$x_0 = \frac{-a+b+c+d}{2} - A \tag{43}$$

Substituting (43) into (41) and (23), we obtain the particular solution

$$\varpi_{mICCI} = \left(a, \frac{-a+b+c+d}{2} - A, \frac{a+b+c-d}{2} + A, d\right) = \varpi_{ICCA} \tag{44}$$

of the equations system (20), (21) and (23). If the sequence ϖ_{mICCI} is monotonic, then the mICCI-approximation task has the unique solution

$$\overleftrightarrow{Tr}_{mICCI}\left(\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)\right) = \overleftrightarrow{Tr}(\varpi_{ICCA}) = \overleftrightarrow{Tr}_{ICCA}\left(\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)\right) \tag{45}$$

This means, inter alia, that in the context of condition (23), the restrictions imposed by condition (19) do not affect the final form of the solution. For this reason, we can opt out of the mICCI-approximation task.

Example 8: For OFN $OFN\ \overleftrightarrow{\mathcal{Y}} = \overleftrightarrow{\mathcal{L}}(3; 2; 1; 0; S_Y; E_Y)$ determined by (40), we have

$$\overleftrightarrow{Tr}_{mICCI}(\overleftrightarrow{\mathcal{Y}}) = \overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{Y}}) = \overleftrightarrow{Tr}(3, {}^{8}/_{3}, {}^{1}/_{3}, 0)$$

The values $a\left(\overleftrightarrow{Tr}_{mICCI}(\overleftrightarrow{\mathcal{V}})\right) = a\left(\overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{V}})\right)$ and $a\left(\overleftrightarrow{Tr}_{mICCI}(\overleftrightarrow{\mathcal{V}})\right) = g\left(\overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{V}})\right)$ are presented in Table 1.□

Table 1. Examples of OFN approximations

OFN $\overleftrightarrow{\mathcal{K}}$	**Value of $\overleftrightarrow{\mathcal{K}}$**	$\boldsymbol{a(\overleftrightarrow{\mathcal{K}})}$	$\boldsymbol{g(\overleftrightarrow{\mathcal{K}})}$
$\overleftrightarrow{\mathcal{V}}$	$\overleftrightarrow{\mathcal{L}}(15,11,6,5, S_V, E_V)$	-7.8362	-4.9003
$\overleftrightarrow{Tr}_{UC}(\overleftrightarrow{\mathcal{V}})$	$\overleftrightarrow{Tr}(15,11,6,5)$	-7.5000	-2.5000
$\overleftrightarrow{Tr}_{CA}(\overleftrightarrow{\mathcal{V}})$	$\overleftrightarrow{Tr}(15.1681, 11.1681, 5.8319, 4.8319)$	-7.8362	-2.5000
$\overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{V}})$	$\overleftrightarrow{Tr}(15, 11.3362, 5.6638, 5)$	-7.8362	-2.1638
$\overleftrightarrow{Tr}_{mCI}(\overleftrightarrow{\mathcal{V}})$	$\overleftrightarrow{Tr}(14{,}5306, 11{,}8056, 5.1944, 3{,}9694)$	-7.8362	-4.9003
$\overleftrightarrow{\mathcal{Y}}$	$\overleftrightarrow{\mathcal{L}}(3,2,1,0, S_Y, E_Y)$	$-{}^{8}/_{3}$	$-{}^{1}/_{3}$
$\overleftrightarrow{Tr}_{ICCA}(\overleftrightarrow{\mathcal{Y}})$	$\overleftrightarrow{Tr}(3, {}^{8}/_{3}, {}^{1}/_{3}, 0)$	$-{}^{8}/_{3}$	$-{}^{1}/_{3}$

$\overleftrightarrow{Tr}_{mICCI}(\overleftrightarrow{\mathcal{Y}})$	$\overleftrightarrow{Tr}(3, {}^{8}/_{3}, {}^{1}/_{3}, 0)$	$-{}^{8}/_{3}$	$-{}^{1}/_{3}$

Source: Own elaboration

Note that omitting the criterion of indistinctness invariance can significantly change the picture of imprecision.

5. Final Remarks

As a result of replacing Kosko's entropy measure (1986) with the entropy measure proposed in (Czogała et al., 1981), one of the OFN approximation methods using TrOFN has changed. This replacement does not cause any additional formal and calculation problems. On the benefit side, however, we note the removal of restrictions imposed by maximal feasible value of Kosko's measure of entropy. In the context of experience gathered in this work and in (Piasecki, Łyczkowska-Hanćkowiak, 2018), it should be stated that the most faithful approximation method is mCI-approximation. In this situation, mCI-approximation should be the first choice method. And only in the absence of a solution designated by this method should be attempted use of successive approximation methods. Then we should follow the guidelines presented in (Piasecki, Łyczkowska-Hanćkowiak, 2018).

Finally, it should be pointed out that the research on the impact of OFN approximation on the portfolio analysis is appropriate.

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