BEHAVIOURAL PRESENT VALUE

DETERMINED BY ORIENTED FUZZY NUMBER

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***Abstract**: The starting point for our discussion is the notion of the Behavioural Present value (BPV) defined with use behavioural premises as a positive fuzzy number (FN). The information described by BPV may be supplemented with a forecast of the asset price closest changes. Then the BPV is called oriented BPV (O-BPV). In the first approach, an O-BPV was described by ordered FNs introduced by Kosiński and his co-operators. For the formal reasons, the Kosiński's theory was revised. Each revised ordered FN is called oriented FN (OFN). The main purpose of our study is to describe O-BPV by OFNs. We consider here six cases of BPV: overvalued asset price with prediction of its rise, overvalued asset price with prediction of its fall, undervalued asset price with prediction of its rise, and undervalued asset price with prediction of its fall. All our considerations are illustrated by means of simple numerical example.*

***Keywords**: behavioural finance, present value, oriented fuzzy number.*
***JEL Classification**: C44, C02, G10*

1. Introduction

Ordered fuzzy numbers are defined by Kosiński et al (2002) who in this way were going to introduce a fuzzy number (FN) supplemented by orientation. For some formal reason (Kosiński, 2006), the original Kosiński's theory was revised in (Piasecki, 2018). At present, the ordered FNs defined within Kosiński's original theory are called Kosiński's numbers (Prokopowicz, 2015; Prokopowicz & Pedrycz, 2015; Piasecki, 2019, Piasecki & Łyczkowska-Hanćkowiak, 2020). If ordered FN is linked to the revised theory, then it is called Oriented FN (OFN) (Piasecki, 2019, Piasecki & Łyczkowska, 2020).

In (Piasecki, 2011; Piasecki & Siwek, 2015) the behavioural present value (BPV) was defined as such approximation of fair price which is under impact of behavioural factors. Then BPV is imprecisely estimated by FN. In (Łyczkowska-Hanćkowiak, 2017) the information described by BPV is supplemented with a forecast of the price trend. The positive orientation of fuzzy number describes a subjective prediction of rise in price. The negative orientation of fuzzy number describes a prediction of fall in price. This forecast was implemented in the model BPV as an orientation of FN. In this way the BPV was replaced by oriented BPV (OBPV) described by a Kosiński number. The next year, the first approach to determining OBPV using OFN was presented (Łyczkowska-Hanćkowiak, 2018). Unfortunately, the OBPV membership function was described there by a logically complicated identity system. This made it very difficult to use OBPV.

In this paper we present a revised approach to OBPV. Our main goal is to simplify the identities describing the OBPV membership function. Here, we use our experience gathered during our work on the other OFN application.

2. Oriented fuzzy numbers – basic facts

The symbol $\mathcal{F}(\mathbb{R})$ denotes the family of all fuzzy subsets in the real line $\mathbb{R}$. Any fuzzy subset $\mathcal{A} \in \mathcal{F}(\mathbb{R})$ is described by its membership function $\mu_A \in [0;1]^{\mathbb{R}}$, as the set of ordered pairs

$$\mathcal{A} = \{(x, \mu_A(x)); x \in \mathbb{R}\}. \tag{1}$$

Among other things, this fuzzy subset $\mathcal{A}$ may be characterized by its support closure $[\mathcal{A}]_{0^+}$ given in a following way:

$$[\mathcal{A}]_{0^+} = \lim_{\alpha \to 0^+} \{x \in \mathbb{X}: \mu_A(x) \geq \alpha\}. \tag{2}$$

A commonly accepted model of imprecise number is the fuzzy number (FN), defined as a fuzzy subset of the real line $\mathbb{R}$. The most general definition of FN is given as follows:

Definition 1 (Dubois & Prade, 1978)**.** *The fuzzy number (FN) is such a fuzzy subset* $\mathcal{L} \in \mathcal{F}(\mathbb{R})$ *with bounded support closure* $[\mathcal{L}]_{0^+}$ *that it is represented by its upper semi-continuous membership function* $\mu_L \in [0;1]^{\mathbb{R}}$ *satisfying the conditions:*

$$\exists_{x \in \mathbb{R}} \ \mu_L(x) = 1 \tag{3}$$

$$\forall_{(x,y,z)\in\mathbb{R}^3} \ x \le y \le z \Longrightarrow \mu_L(y) \ge \min\{\mu_L(x); \mu_L(z)\}. \tag{4}$$

The set of all FN we denote by the symbol $\mathbb{F}$.

Thanks to the results obtained in (Goetschel & Voxman, 1986), we have that any FN can be equivalently defined as follows:

Theorem 1 (Delgado et al, 1998) *For any FN $\mathcal{L}$ there exists such a non-decreasing sequence $(a,b,c,d) \subset \mathbb{R}$ that $\mathcal{L}(a,b,c,d,L_L,R_L) = \mathcal{L} \in \mathcal{F}(\mathbb{R})$ is determined by its membership function $\mu_L(\cdot\,|a,b,c,d,L_L,R_L) \in [0,1]^{\mathbb{R}}$ described by the identity*

$$\mu_L(x|a,b,c,d,L_L,R_L) = \begin{cases} 0, & x \notin [a,d], \\ L_L(x), & x \in [a,b[, \\ 1, & x \in [b,c], \\ R_L(x), & x \in]c,d], \end{cases} \tag{5}$$

where the left reference function $L_L \in [0,1[^{[a,b[}$ and the right reference function $R_L \in [0,1[^{]c,d]}$ are upper semi-continuous monotonic ones meeting the condition:

$$[\mathcal{L}]_{0^+} = [a,d]\,. \tag{6}$$

The FN $\mathcal{L}(a,a,a,a,L_L,R_L) = [\![a]\!]$ represents the real number $a \in \mathbb{R}$. Therefore, we can say $\mathbb{R} \subset \mathbb{F}$. For any $z \in [b,c]$, a FN $\mathcal{L}(a,b,c,d,L_L,R_L)$ is a formal model of linguistic variable "about z". Understanding the phrase "about z" depends on the applied pragmatics of the natural language.

The notion of ordered FN is intuitively introduced by Kosiński et al (2002), as such model of imprecise number that subtraction of ordered FNs is the inverse operator to their addition. Therefore, OFNs can contribute to specific problems concerning the solution of fuzzy linear equations of the form or help with the interpretation of specific improper fuzzy arithmetic results. Currently, ordered FNs defined in this way are called Kosiński's numbers (Prokopowicz, 2015; Prokopowicz & Pedrycz, 2015; Piasecki, 2019, Piasecki & Łyczkowska-Hanćkowiak, 2020).

An important disadvantage of Kosiński's theory is that there exist such ordered FNs which are not linked to any membership function (Kosiński, 2006). For this reason, the Kosiński's theory is revised in (Piasecki, 2018). In revised theory, Kosiński's numbers are replaced by OFNs defined as follows:

Definition 2 (Piasecki, 2018) *For any monotonic sequence* $(a, b, c, d) \subset \mathbb{R}$, *the oriented fuzzy number OFN* $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L) = \overleftrightarrow{\mathcal{L}}$ *is the pair of orientation* $\overrightarrow{a, d} = (a, d)$ *and fuzzy set* $\mathcal{L} \in \mathcal{F}(\mathbb{R})$ *described by membership function* $\mu_L(\cdot\,|a, b, c, d, S_L, E_L) \in [0,1]^{\mathbb{R}}$ *given by the identity*

$$\mu_L(x|a, b, c, d, S_L, E_L) = \begin{cases} 0, & x \notin [a, d] \equiv [d, a], \\ S_L(x), & x \in [a, b[\, \equiv\,]b, a], \\ 1, & x \in [b, c] \equiv [c, b], \\ E_L(x), & x \in\,]c, d] \equiv [d, c[. \end{cases} \tag{7}$$

where the starting function $S_L \in [0,1[^{[a,b[}$ *and the ending function* $E_L \in [0,1[^{]c,d]}$ *are upper semi-continuous monotonic ones meeting the condition (6).*

The identity (7) additionally describes such modified notation of numerical intervals which is applied in this work.

The symbol $\mathbb{K}$ denotes the space of all OFNs. Any OFN describes an imprecise number with additional information about the location of the approximated number. This information is given as orientation of OFN. If $a < d$ then OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ has the positive orientation $\overrightarrow{a, d}$. In this case, the starting-function S_L is non-decreasing and the ending-function E_L is non-increasing. For any $z \in [b, c]$, the positively oriented OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is a formal model of linguistic variable “about or slightly above z”. Therefore, any positively oriented OFN may be interpreted as imprecise number, which may increase.The symbol $\mathbb{K}^+$ denotes the space of all positively oriented OFN. If $a > d$, then OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ has the negative orientation $\overrightarrow{a, d}$. In this case, the starting-function S_L is non-increasing and the ending-function E_L is non-decreasing. For any $z \in [c, b]$, the negatively oriented TrOFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is a formal model of linguistic variable “about or slightly below z”. For this reason, negatively oriented OFN is interpreted as an imprecise number, which may decrease. The symbol $\mathbb{K}^-$ denotes the space of all negatively oriented OFN. Understanding the phrases “about or slightly above z” and “about or slightly below z“ depend on the applied pragmatics of the natural language. If $a = d$, OFN $\overleftrightarrow{\mathcal{L}}(a, a, a, a, S_L, E_L) = [\![a]\!]$ describes unoriented number $a \in \mathbb{R}$.

3. Oriented fuzzy present value

The present value (PV) is defined as a present equivalent of a cash flow in a given time in the present or future (Piasecki, 2012). It is commonly accepted that the PV of a future cash flow can be imprecise. The natural consequence of this approach is estimating PV with FNs. Such

PV is called fuzzy one. A detailed description of the evolution of this particular model can be found in (Piasecki, 2014). In general, fuzzy PV is characterized by non-decreasing sequence $\{V_s, V_f, \check{P}, V_l, V_e\}$, where:

- $\check{P}$ - price,
- $[V_s, V_e] \subset \mathbb{R}^+$ is interval of all possible PV values,
- $[V_f, V_l] \subset [V_s, V_e]$ is interval of all prices which do not perceptible differ from price $\check{C}$.

Then fuzzy PV is estimated by FN

$$\widetilde{PV} = \mathcal{L}(V_s, V_f, V_l, V_e, L_{PV}, R_{PV}), \tag{8}$$

where the left reference functions $L_{PV} \in [0,1[^{[V_s,V_f[}$ and $R_{PV} \in [0,1[^{]V_l,V_e]}$ are given ones.

Moreover, fuzzy PV estimation should be supplemented by forecast of price closest changes. For example, price closes changes may be predicted with use prediction table presented in (Piasecki & Stasiak, 2019). For these reasons, imprecise PV may be evaluated by OFN (Łyczkowska-Hanćkowiak, 2019; Piasecki & Łyczkowska-Hanćkowiak, 2020). Such PV is called oriented PV (OPV). Any OPV is characterized by monotonic sequence $\{V_s, V_f, \check{P}, V_l, V_e\}$ and then it is estimated by OFN

$$\overleftrightarrow{PV} = \overleftrightarrow{\mathcal{L}}(V_s, V_f, V_l, V_e, L_{PV}, R_{PV}), \tag{9}$$

If we predict a rise in price then OPV is described by positively oriented OFN. If we predict a fall in price, then OPV is described by negatively oriented OFN.

4. Oriented fuzzy behavioural present value

Let us consider any financial asset which is the subject of trade on the highly effective financial market. The price $\check{P}$ of this asset may fluctuate over time. Therefore, we can consider a price trend. In technical analysis, we always assume that in the nearest short time period this trend has a fixed point P_0 called the balanced price. Current value of balanced price P_0 is substantively justified by fundamental analysis. Then we conclude that the price $\check{P}$ should converge to the balanced price P_0. If the condition $\check{P} > P_0$, then considered asset is overvalued. For the case $\check{P} < P_0$, considered asset is undervalued. Both of these cases we call the state of financial disequilibrium. Financial equilibrium is state in which asset prices satisfy the condition $\check{P} = P_0$. Then considered asset is fully valued. The state of financial equilibrium/disequilibrium is characterized by deviation of the price $\check{P}$ from the balanced price P_0 determined as follows

$$\Delta P = \check{P} - P_0. \tag{10}$$

The relative deviation of price $\check{P}$ from the balanced price P_0 is given by the formula

$$\delta P = \frac{|\Delta P|}{\check{P}}. \tag{11}$$

In (Piasecki, 2012) was defined PV as the utility function of financial flow. This PV depends on both subjective and objective premises. Subjective premises are behavioural in nature. In (Piasecki, 2011; Piasecki & Siwek, 2015), behavioural PV (BPV) is originally introduced as PV depending on selected behavioural factors. An image of behavioural factors is always imprecise. Hence PV deviation from the current price $\check{P}$ is imprecise.

Let considered asset be characterized by the vector $\boldsymbol{v} = \left(\check{P}, P_0, V_{min}, V_{max}\right)$, where

- V_{min} – the greatest lower bound of the PV evaluation,
- V_{max} – the least upper bound of the PV evaluation.

In (Piasecki, 2011; Piasecki & Siwek, 2015), we can find some method determining the bounds of PV evaluation. Moreover, there is justified that BPV is determined as FN $\widetilde{BPV}(\boldsymbol{v})$ given by its membership function $\mu_{BPV}(\cdot|\boldsymbol{v}) \in [0;1]^{\mathbb{R}^+}$ in following way

$$\mu_{BPV}(x|\boldsymbol{v}) = \begin{cases} 0, & x \notin [V_{min}, V_{max}], \\ h(x|\boldsymbol{v}), & x \in \left[V_{min}, \check{P}\right[, \\ 1, & x \in \left[\check{P}, \check{P}\right], \\ k(x|\boldsymbol{v}), & x \in \left]\check{P}, V_{max}\right], \end{cases} \tag{12}$$

where the functions $h(\cdot|\boldsymbol{v}) \in [0,1[^{[V_{min},\check{P}[}$ and $k(\cdot|\boldsymbol{v}) \in [0,1[^{]\check{P},V_{max}]}$ are determined by the identities

$$h(x|\boldsymbol{v}) = \begin{cases} \frac{(x-V_{min})(1+\delta P)}{\check{P}-V_{min}+(x-V_{min})\delta P} & \Delta P > 0 \\ \frac{x-V_{min}}{\check{P}-V_{min}+(x-V_{min})\delta P} & \Delta P \leq 0 \end{cases}, \tag{13}$$

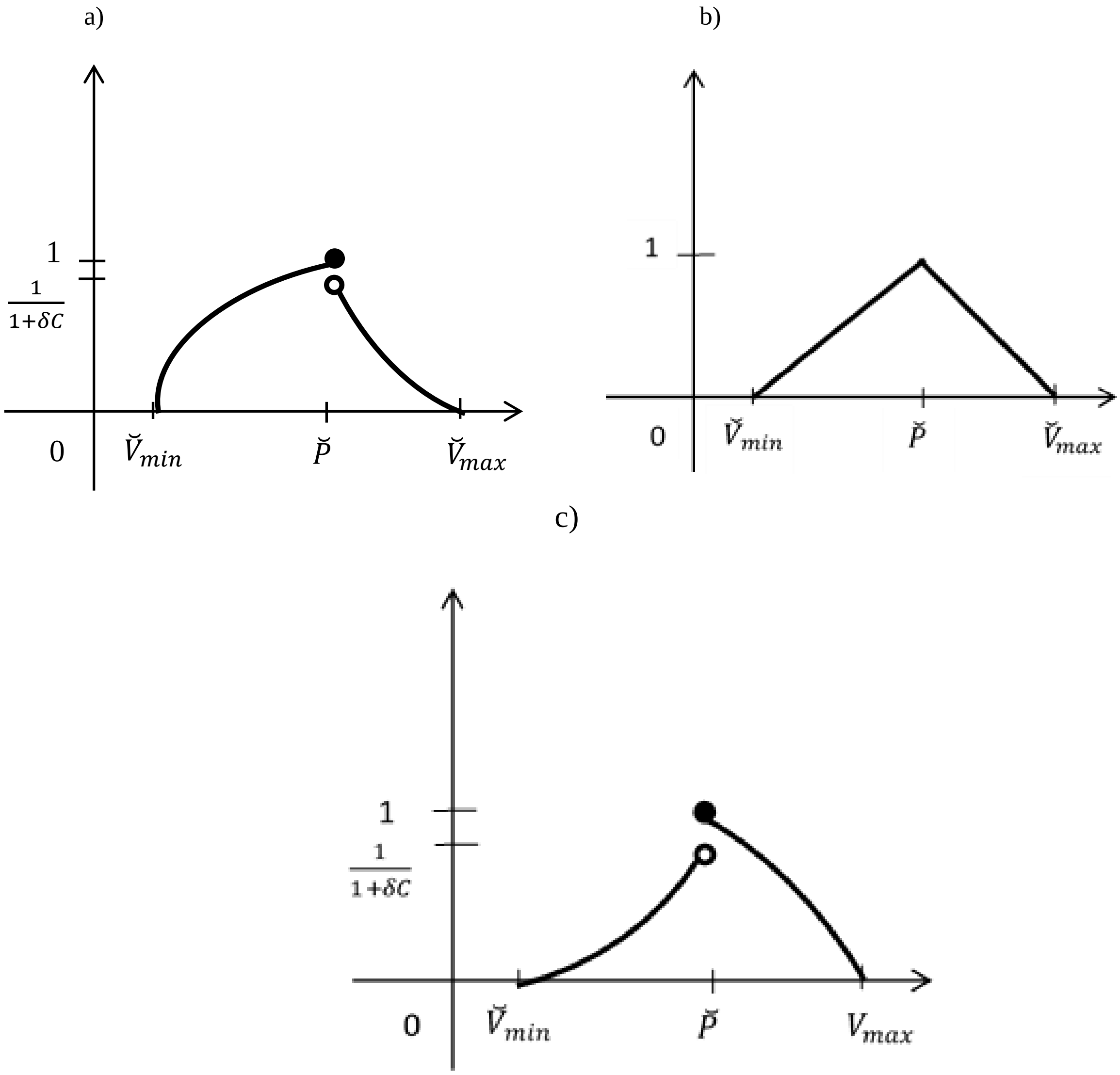
$$k(x|\boldsymbol{v}) = \begin{cases} \frac{V_{max}-x}{V_{max}-\check{P}+(V_{max}-x)\delta P} & \Delta P > 0 \\ \frac{(V_{max}-x)(1+\delta P)}{V_{max}-\check{P}+(V_{max}-x)\delta P} & \Delta P \leq 0 \end{cases}. \tag{14}$$

In general, BPV is FN

$$BPV = \mathcal{L}\left(V_{min}, \check{P}, \check{P}, V_{max}, h, k\right) \tag{15}$$

which is an approximation of the price $\check{P}$. BPV is described by its membership function μ_{BPV} determined separately for overvalued assets fulfilling the condition $\Delta P > 0$, fully valued assets fulfilling the condition $\Delta P = 0$, and undervalued assets fulfilling the condition $\Delta P < 0$. Figure 1 shows a graphs of these membership functions.

Fig. 1 A graphs of membership function of BPV a) for overvalued assets ($\Delta P > 0$), b) for fully valued assets ($\Delta P = 0$), c) for undervalued assets ($\Delta P < 0$),



Source: Own elaboration

If we take into account predictions of price closest changes, we substitute BPV by oriented BPV (OBPV) given by OFN

$$\overleftrightarrow{BPV} = \overleftrightarrow{\mathcal{L}}(V_s, \breve{P}, \breve{P}, V_e, L_{BPV}, R_{BPV}) \tag{16}$$

where

- $[V_s, V_e] = [V_{min}, V_{max}] \equiv [V_{max}, V_{min}] \subset \mathbb{R}^+$ is interval of all possible BPV' values,
- (L_{BPV}, R_{BPV}) is any ordered pair of starting-function and ending-function.

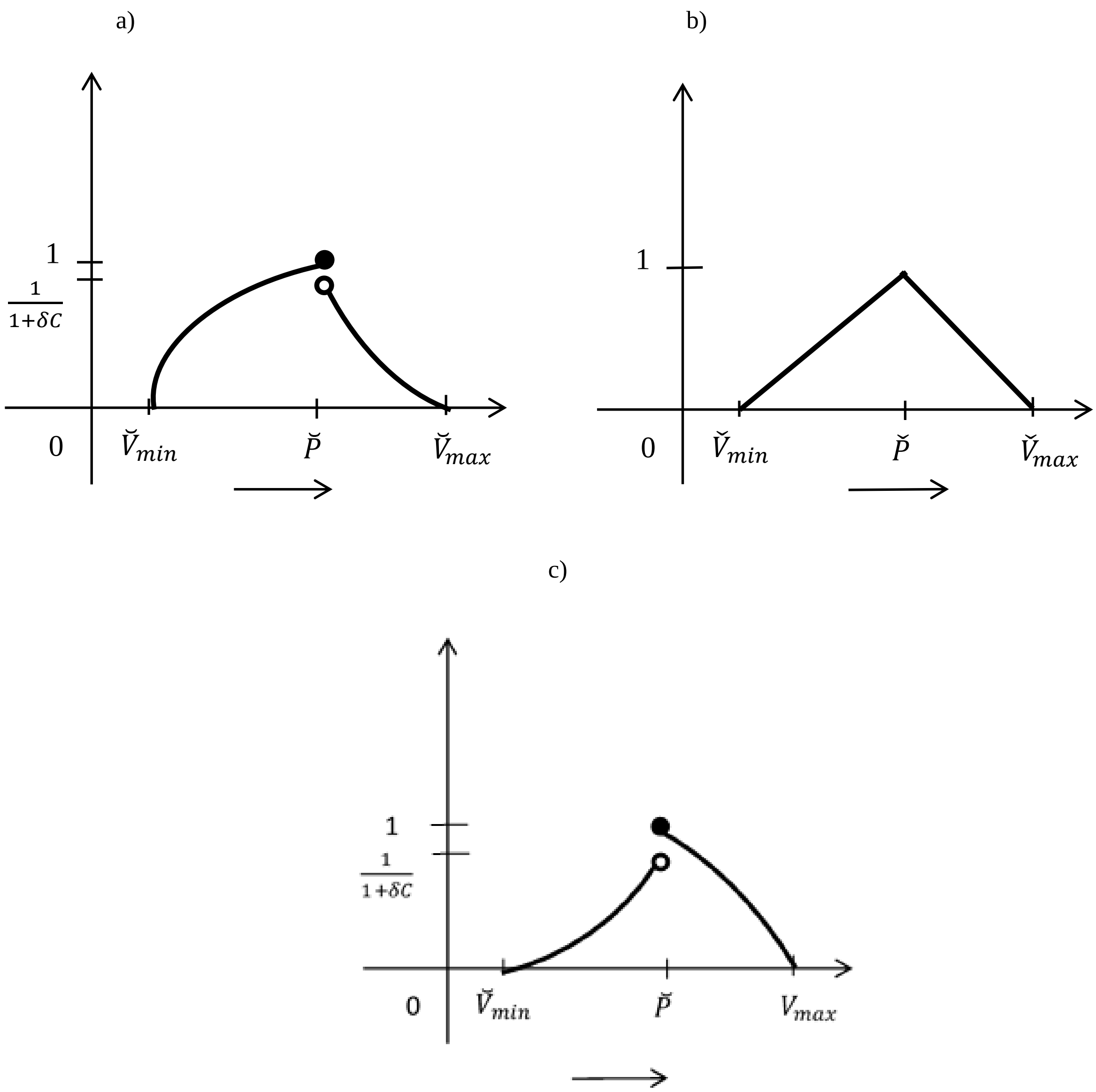
The OBPV $\overleftrightarrow{BPV}$ is determined by its membership function $\mu_{BPV}(\cdot\,|\boldsymbol{v})$ given by the identity (12).

The forecast of price increase is described by the positive orientation of OBPV. Then we have

$$\overleftrightarrow{BPV} = \overleftrightarrow{\mathcal{L}}\left(V_{min}, \check{P}, \check{P}, V_{max}, h, k\right). \tag{17}$$

In this way, we obtain three cases of OBPV predicting rise in asset price: for overvalued assets, for fully valued assets, and for undervalued assets. The membership functions of these OBPV kinds are presented on Fig. 2.

Fig. 2 A graphs of membership function of OBPV predicting rise in price a) for overvalued assets ($\Delta P > 0$), b) for fully valued assets ($\Delta P = 0$), c) for undervalued assets ($\Delta P < 0$),



Source: Own elaboration

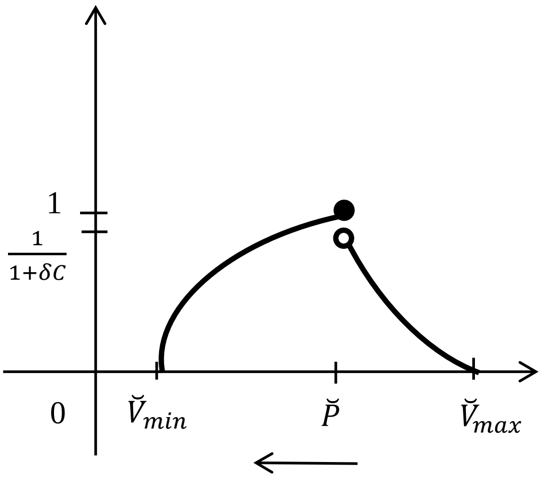
The forecast of price decrease is described by the negative orientation of OBPV. Then we have

$$\overleftrightarrow{BPV} = \overleftrightarrow{\mathcal{L}}\left(V_{max}, \breve{P}, \breve{P}, V_{min}, k, h\right). \tag{18}$$

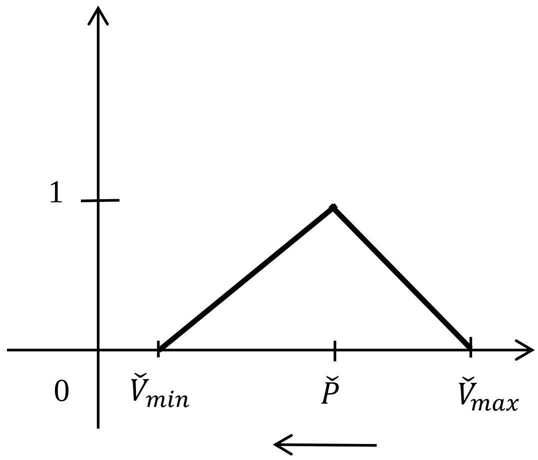
In this way, we obtain three cases of OBPV predicting fall in asset price: for overvalued assets, for fully valued assets, and for undervalued assets. The membership functions of these OBPV kinds are presented on Fig. 3.

Fig. 3 A graphs of membership function of OBPV predicting fall in price a) for overvalued assets ($\Delta P > 0$), b) for fully valued assets ($\Delta P = 0$), c) for undervalued assets ($\Delta P < 0$),

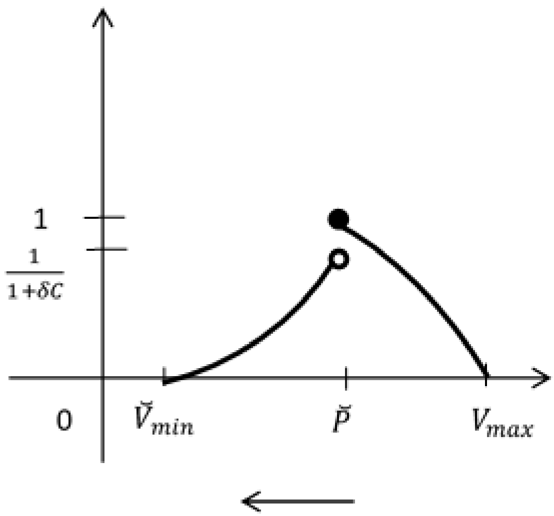
a)



b)



c)



Source: Own elaboration

Example 1. For considered security we observe price $\check{P} = 60$. Substantially justified its balanced price is given as follows $P_0 = 40$. The greatest lower and the least upper bounds of PV evaluation are respectively $V_{min} = 30$ and $V_{max} = 80$. We have

$$\Delta P = \check{P} - P_0 = 60 - 40 = 20 > 0 \,,$$

$$\delta P = \frac{|\Delta P|}{\check{P}} = \frac{20}{60} = \frac{1}{3}.$$

Additionally, we predict fall in price of considered security. Then OBPV is described as negatively oriented OFN

$$\overleftrightarrow{BPV} = \overleftrightarrow{\mathcal{L}}(80{,}60{,}60{,}30, k, h),$$

where

$$h(x) = \frac{4x-120}{x+60} \qquad \text{for} \quad x \in \,]60{,}30],$$

$$k(x) = \frac{3x-240}{x-140} \qquad \text{for } x \in [80; 60[.$$

It means that considered OBPV is explicitly determined by its membership function

$$\mu_{BPV}(x|60{,}40{,}80{,}30) = \begin{cases} \dfrac{3x-240}{x-140} & for\ x \in [80; 60[\\ 1 & for\ x \in [60{,}60] \\ \dfrac{4x-120}{x+60} & for\ \ x \in \,]60{,}30] \\ 0 & for\ x \notin [80{,}30] \end{cases}$$

□

6. Conclusions

If we compare the results obtained with the results presented in (Łyczkowska-Hanćkowiak, 2018), we can conclude that the identities determining the OBPV membership function has been simplified. Given in the article objective has been achieved.

This will facilitate the use of OBPV for an analysis of financial instruments with imprecise estimated values. For example, obtained results may be applied in invest decision models described in (Piasecki, 2014). It is expedient to further development the fuzzy finance theory based on OFN.

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